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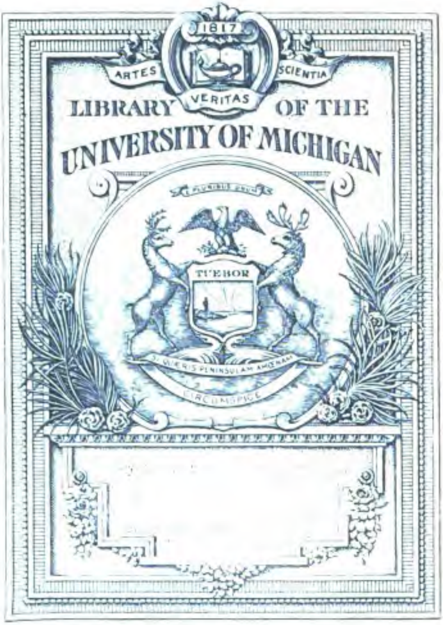

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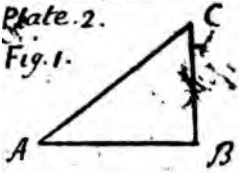
Plate. 2.

Fig. 1.



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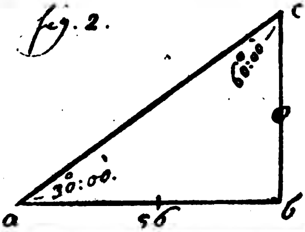

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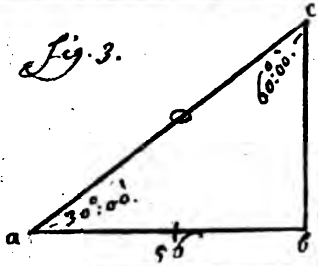

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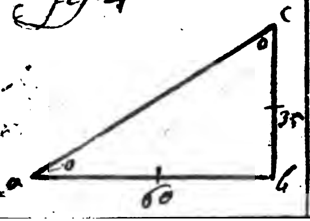

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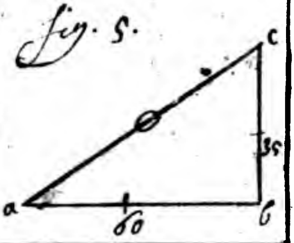

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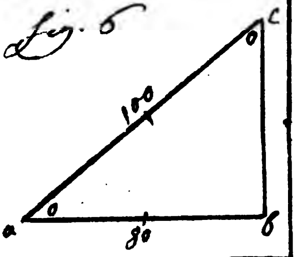

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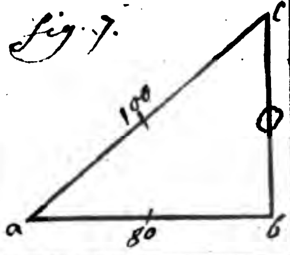

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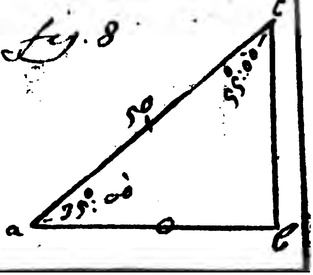

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A TREATISE OF TRIGONOMETRY,

PLANE and SPHERICAL, THEORETICAL and PRACTICAL.

In which the ſeveral Caſes of *Plane* and *Spherical* Triangles are ſolved, *Inſtrumentally* and *Arithmetically*.

As likewiſe a TREATISE OF STEREOGRAPHICK and ORTHOGRAPHICK Projection of the SPHERE.

In which the *Principles* and *Theorems* on which they depend, are clearly Demonſtrated, and the Practice naturally deduced from thoſe Demonſtrations.

Illuſtrated in the Stereographick Projection of the ſeveral Caſes in Right and Oblique Angled, Spherical Triangles. So that the Requiſites may be found without Calculation, by Scale and Compaſs.

The Second Edition, carefully corrected by the Author
SAM^{uel}. HEYNES, *Teacher of the*
MATHEMATICKS.

LONDON,
Printed for *R.* and *W. Mount*, and *T. Page*, in *Poſtern Row*, on *Tower-Hill*. MDCCXVI. 1716

An Explanation of the Symbols us'd in this Treatiſe.

$+$	Signifies	Addition.
$-$		Subtraction, *or the difference of the two Quantities between which 'tis plac'd.*
$\backsim$		The Difference *of two Quanties when it is not known in which the Exceſs lies.*
$\times$		Multiplication.
$>$		Majority, *or greater than.*
$<$		Minority, *or leſſer than.*
$=$		Equality.
$\surd$		Radicality, *or Square Root.*
$\square$		Rectangle, *or Product.*
R		Radius.
S		Sine.
Σ		Co-ſine.
ſ		Secant.
σ		Co-ſecant.
T		Tangent.
τ		Co-tangent.
V		Verſed-ſine.
v		Verſed-ſine *of the Supplement.*
L		Right-angle.
Z *cr*		Sum *of the Legs.*
X *cr*		Difference *of the Legs.*
Z *L L*		Sum *of the Angles.*
X *L L*		Difference *of the Angles.*

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TO

Capt. *Edm. Halley*,

Fellow of the ROYAL SOCIETY, &c.

SIR,

THIS Treatiſe of *Trigonometry* and *Stereographick Projection* is humbly Dedicated to You; I being glad of this Opportunity of more Publickly Teſtifying the Reſpect I bear to You, and my Obligations for the Kindneſſes conferr'd by You upon

Your Humble and
Oblig'd Servant

SAM. HEYNES.

TO THE

READER.

THis Treatiſe is accommodated both to the Theoretick and Practick Reader; a Practical Treatiſe only, would never pleaſe the former, nor a Theorick the latter.

In order to the former End I have made uſe of the moſt Conciſe and Elegant Demonſtrations I could find, and particularly from that incomparable Treatiſe of Trigonometry; Written by the Learned Mr. Caſwell, *A. M. of* Oxford.

And that I might anſwer the latter End the better, I have given the actual Solutions of the ſeveral Caſes both of Plane and Spherical Triangles in Numbers, Natural and Artificial, and likewiſe by the Gunter's Scale; and have taken great care that the ſeveral Directions to practice ſhould be ſo plain and eaſy, that they might be made uſe of effectually by Perſons that are not ſo well grounded as yet in Geometry, &c. *as to underſtand the Demonſtrations.*

As to the other Treatiſe of Sterographick Projection *of the* Sphere, *which is a Branch of Perſpective; I have Demonſtrated the Principles and Theorems on which it depends, and from thoſe Theorems have ſhew'd the reaſon of the ſeveral Methods made uſe of in Practice, making thereby the Projection of the Sphere upon the Plane of any great Circle to become eaſy, and likewiſe the Menſuration of any are ſo Projected.*

And laſtly, I have ſhew'd you how to Project the ſeveral Caſes of Spherical Triangles both Right and Oblique, the Reaſons of all which Precepts naturally flowing from Preceeding Propoſitions: There hath been nothing wrote in Engliſh *giving ſo full Account of this Matter, which is humbly Submitted to the Judicious Reader. By*

S. *H*

SOME FEW Geometrical Problems,

Neceſſary to be known in Order to find the Requiſits of Right and Oblique Triangles, *by a* Scale *of Equal* Parts, *and* a Line *of* Chords.

PROBLEM I.

From a Point in a Line given, to erect a Perpendicular.

LET the Line given be *a b*, and the Point given *e*; take two Points *c*, *d*, equally diſtant from *e*: Then putting the Foot of your Compaſſes in *c* and *d*, open'd to any diſtance greater than *c e*, and ſtrike two Arches cutting one another in *f*. Then draw the Line *f e*, it ſhall be the Perpendicular required

Plate I. *Fig.* I.

PROB. II.

To divide a given Right-line into two equal Parts.

Let the Line given be *a b*; upon the Point *a*, with your Compaſſes open'd to any Diſtance greater than half the Line *a b*, diſcrbe the Arch *c d*; then putting the Foot of your Compaſſes in the Point *b*, and with the ſame opening de-

Plate I. *Fig.* II.

ſcribe an Arch cutting the other in *c* and *d*, the Line *c d* ſhall biſſect the Line given in *e*.

PROB. III.

From a Point given without to let fall a Perpendicular to a Line given.

Plate I. Fig. III.

Let the Point given be *g*, and the Line given *a b*; put one Foot of your Compaſſes in *g*, and with any Diſtance, greater than the neareſt Diſtance of that Point from the Line given, croſs the Line given in the Points *c* and *d*, then putting the Foot of your Compaſſes on the Points *c* and *d*, with the ſame openning of your Compaſſes deſcribe two Arches cutting one another in *f*, a Rular being laid to *g* and *f*, draw the Line *g b*, it ſhall be the Perpendicular requir'd.

PROB. IV.

From the end of a Line given, to erect a Perpendicular.

Plate I. Fig. IV.

1. Upon the Point *b* with your Compaſſes open'd to any Diſtance *b c*, deſcribe the Arch *c d e*, then your Compaſſes at this opening will reach from *c* to *d*, and from *d* to *e*, upon the Points *d* and *e*, and with the ſame opening of your Compaſſes deſcribe two Arches cutting one another in *f*; *f b* ſhall be the Perpendicular.

Plate I. Fig. V.

2. Your Compaſſes being open'd at any Diſtance, put one Foot in *b*, and the other any where in *e*, then turn about the Foot that was in *b*, till it croſs the Line *a b* given in *d*, draw the Line *d e f*, ſo that *e f* = *e d*, then is *f b* a Perpendicular.

Fig: I.

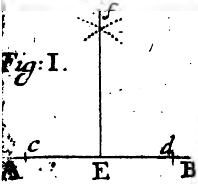


Fig: II.

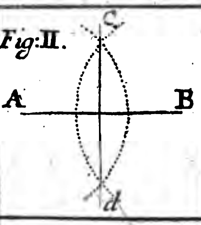


Fig: III.

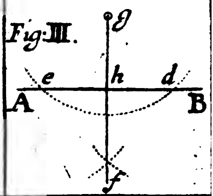


Fig: IIII.

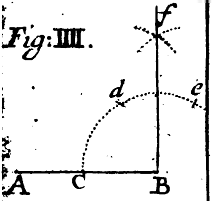


Fig: V.

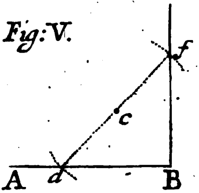


Fig: VI.

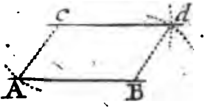


Fig: VII.

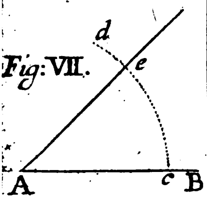


Fig: VIII.

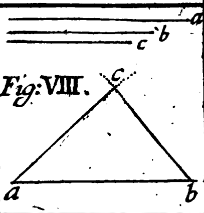


Fig: IX

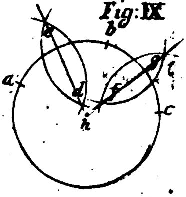


Fig: X.

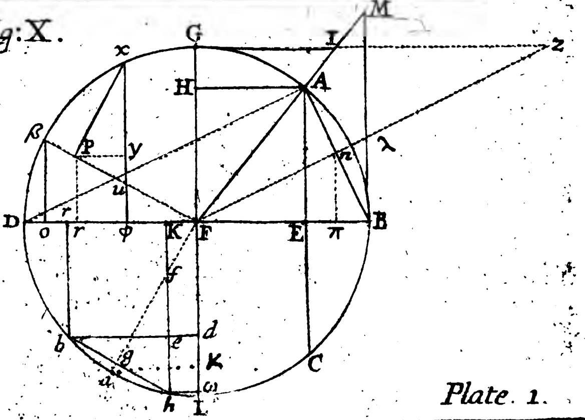


Plate. 1.

PROB. V.

From a Point given to draw a Parrallel to a Line given.

Let the Point given be *c*, and the Line given *a b*; From the Point *c* as a Center, ſtrike an Arch cutting the given Line *a b* in *a*, then take *a c* in your Compaſſes, and putting one Point any where in the Line *a b*, ſuppoſe in *b*, with the other deſcribe an Arch, then take *a b* in your Compaſſes, and putting one Foot in *c*, with the other cut the former Arch in *d*, a Line drawn thro' *c* and *d* is Parrallel to the Line given.

Plate I.
Fig. VI.

PROB. VI.

With a Line given a b, *to make an Angle of any Number of Degrees given, Suppoſe* 45 *Degrees.*

Open your Compaſſes to 60 *deg.* of your Line of Chords (60 *deg.* of the Chords being equal to the Radius of that Circle, the Chords were deſign'd to meaſure) and putting one Foot in *a*, with the other deſcribe the Arch *e d*; then take 45 *deg.* from your Scale of Chords, and put it from *e* to *e* in the Arch, draw the Line *a e*, and 'tis done.

Plate I.
Fig. VII.

Hence an Angle being already made, you may meaſure its Quantity by taking *e e* in your Compaſſes, and applying it to your Line of Chords and it ſhews you the Number of *deg.* in that Arch that is the Meaſure of the Angle.

PROB. VII.

Three Sides given to make a Triangle.

Let the Sides given be *a*, *b* and *c*, with which it is requir'd to make a Triangle, make *a b*=the Side *a*, then taking the Side *b* in your Compaſſes, with one Foot in *a* deſcribe an Arch, and with the Diſtance *c* putting any Foot in *b*, cut the former Arch in *c*, then drawing *a c* and *b c* and 'tis done.

Plate I. *Fig* VIII.

PROB. VIII.

To bring any three Points not ſituate in a Right-line, into the Circumference of a Circle

Let the three points given be *a b c*, take above half the Diſtanse between *a* and *b* in the Compaſſes, then placing one Foot of your Compaſſes in *a*, with the ſaid Diſtance deſcribe the Arch *e d*, with the ſame Diſtance and one Foot in *b* deſcribe the Arch *e d*, cutting the former in *e* and *d*, and draw the Line *e d*: Then taking above half the Diſtance between *b*, *c*, and one Foot being placed in *c* deſcribe the Arch *f*, *g*, and with the ſame Diſtance, one Foot being placed in the Point *b*, croſs the former Arch in the Points *f* and *g*, and draw the Line *f g*; now where the two Right-lines *d e* and *f g*, Interſect each other (*viz*. in the Point *h*) is the Center of the Circle, which was required.

Plate I. *Fig*. IX.

DEFINITIONS.

1. A Circle is ſuppos'd to be divded into 360 equal Parts call'd Degrees, and each Degree into 60 Parts call'd Minuites, and each Minuite into 60 Parts call'd Seconds, &c. any Portion of which Circumference is call'd an Arch, and is Meaſur'd by the Number of Degrees, it contains.

Plate I. *Fig*. X.

2. A Chord is a Right-line joyning the Extremities of an Arch, as AC is the Chord of the Arches A B C, A D C.

3. A Right-ſine is a Right-line drawn from one end of an Arch, Perpendicular to that Diameter paſſing thro' the other end; or it is half the Chord of twice the Arch, A E is the Right-ſine of the Arches AB, AD: and here 'tis evident, that the Sine of 90 *deg.* (which is equal to the Radius or Semidiameter of the Circle) is the greateſt of all Sines, the Sine of an Arch greater than a Quadrant, being leſs than the Radius.

4. A Verſed-ſine is the Segment of the Diameter intercepted between the Arch and its Right-ſine, E B is the Verſed-ſine of the Arch A B, and E D of the Arch A D.

5. A Tangent of an Arch is a Right-line drawn Perpendicular to the end of a Diameter, paſſing thro' one end of an Arch, and its Length is limited by a Right-line drawn from the Center thro' the other end of the Arch, and this Line is call'd the Secant; thus BM is the Tangent and F M the Secant of the Arches A B, A D.

6. The difference of an Arch from a Quadrant, whether it be greater or leſs is call'd its Complement, GA is the Complement of the Arches A B, AD; HA is the Sine of that Complement or Co-ſine, G I the Tangent of that Complement, or Co-tangent, F I the Secant of that Complement or Co-ſecant.

7. The difference of an Arch from a Semicircle is call'd its Suppliment.

8. That Part of the Radius which is betwixt the Center and Right-ſine is = to the Co-ſine, FE = HA.

9. If an Arch be greater or leſſer than a Quadrant the $\left\{\begin{matrix}\text{Sum}\\ \text{Diff.}\end{matrix}\right\}$ of the Radius and Co-ſine is equal to the Verſed-ſine. Theſe things will be farther Illuſtrated by the following Figure.

In a Triangle are ſix Parts, *viz.* three ſides and three Angles, any three of which being given (except the three Angles of a Plane Triangle) the other three may be found, either *Mechannically* by the help of a Scale of equal Parts and Line of Chords, or by an *Arithmetick* Calculation (if ſuppoſing the Radius divided into any Number of equal Parts we know how many of thoſe equal Parts are in the Sine, Tangent, or Secant of any Arch propos'd) the Art of inferring which is call'd *Trigonometry*, and is either Plane or Sphærical.

The Table of Natural Sines, Tangents, &c. may be thus made,

Fig. 1. Pl. 1.

1. A E the Sine of an Arch being given to find its Co-ſine HA,
$FAq = AEq + FEq$, therefore $\sqrt{FAq - AEq} = FE = HA$, or $\sqrt{Rq - Sq} = \Sigma$.

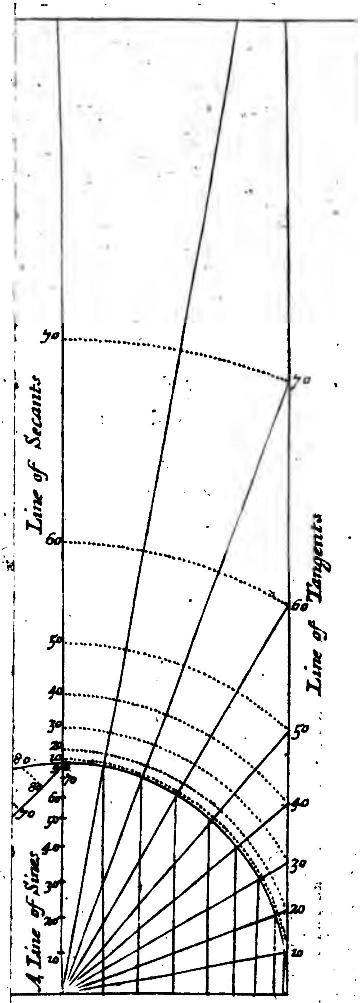
2. A E the Sine of an Arch being given, to find B N the Sine of $\frac{1}{2}$ the Arch.
F E the Co-ſine is known by (the firſt) and conſequently E B; then $\sqrt{AEq + EBq} = AB$, and $\frac{1}{2} AB = BN$ *i. e.* $\frac{1}{2}\sqrt{Sq + Vq} = S\,\frac{1}{2}$ Arch.

3. BN the Sine of an Arch being given, to find A E the Sine of twice that Arch.
FN the Co-ſine is known (by the firſt) and the Triangles F B N, A B E are Similar; therefore FB, FN : : AB. AE. *i.e.* R. Σ : : 2 S. S 2 Arch.

4. βo and xP, the Sines of two Arches βD, $x\beta$ being given; to find $x\varphi$ the Sine of the Sum of the Arches.

Cons. Draw Pe parallel to βo, & Py parallel to DF.

The


Line of Secants
Line of Tangents
A Line of Sines

Page 6.

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The Co-ſines FO. FP are known (*per* firſt) then $F\beta . FP :: \beta o . Pr$. and the Triangles yPx (Pux, $\phi u F$) $o \beta F$ are Similar, therefore $F\beta . Fo :: xP . xy$. then $xy + Pr = x\phi = \frac{Fo \times xP + FP \times \beta o}{F\beta}$ that is the Sine of the one Arch into the others Co-ſine + Sine of the other Arch into the Co-ſine of the firſt Arch, and this Sum divided by the Radius = S of the Sum of two Arches; and if the Sines of the ſame two Arches were given, and it were requir'd to find the Sine of their Difference, it would be $\frac{Fo \times xP \backsim FP \times \beta o}{R} =$ S. of the difference of two Arches.

5. Suppoſe the Arch La = 30 *deg.* and *ab* = *ab*, and *bω* the Sine of *b*L, and *b*K its Co-ſine, alſo *bd* the Sine of *b*L, and *br* its Co-ſine; draw F*a* cutting *bb* in *g*, then $4\,gb\,q$ ($4\,eb, q$) $= bb, q = be, q + eb, q$; therefore $3\,gb, q = be, q$, and $gb \times \sqrt{3} = be$, that is $gb \times 1.732050807 = be$; and $b\omega + gb \times 1.732050807 = bd$, *i. e.* the Sine of an Arch leſs than 30 *deg.* adding $\sqrt{3} \times$ S of the Defect, makes the Sine of an Arch as much exceeding 30 *deg.* therefore if the Sines of all Arches leſs than 30 *deg.* be known, the reſt as far as 60 *deg.* may be had by one Addition, and a Multiplication into the Root of 3.

6. The Triangles *gbf* and *beb* are Similar, but the Angle $hfg = g$FL = 30 *deg.* therefore the Angle *bbe* = 30 *deg.* and ſuppoſing a Circle on the Center *g*, deſcribed through *heb*, *be* the Chord of 60 will be equal to *bg* the Radius, therefore, $rb + bg =$ K*b*, *i. e.* the Sine of any Arch *b*D leſs than 60 *deg.* adding the Sine of the Defect *ba*, make K*b* the Sine of an Arch as much exceeding 60 *deg.*

Conſequently, the Sines of all Arches leſs than 60 *deg.*

being known, the reſt to 90 *deg.* may be found by one Addition. For Inſtance.

S, 58 *deg.* + S, 2 *deg.* = S, 62 *deg.*
S, 50 *deg.* + S, 10 *deg.* = S, 70 *deg.*

An Application of what has been ſaid, to the actual making the aforeſaid Table.

The Radius is equal to the Chord of 60 *deg.* and ½ R = S, 30. then by the Second are known the Sines of the ½ Arches.

deg.	′	″	‴	iv	v
15	00	00	00	00	00
7	30	00	00	00	00
3	45	00	00	00	00
1	52	30	00	00	00
	56	15	00	00	00
	28	07	30	00	00
45	14	03	45	00	00
	07	01	52	30	00
	03	30	56	15	00
	01	45	28	07	30
		52	44	03	45

So that by 12 Diviſions we come to Sines, which have the ſame ſenſible Proportion as their Arches, for the laſt Sine ſave one, is double of the laſt Sine to all Senſe, as one Arch is double of the other.

Now the Number of Minutes in 30 *deg.* is 1800, and $52^{''}.\ 44^{'''}.\ 03^{iv}.\ 45^{v}.$ is contained in 1800 Minutes 2048 times, therefore $1^{'} \times 1800 = 2048 \times 52^{''}.\ 44^{'''}.\ 03^{iv}.\ 45^{v}.$ therefore 1800. 2048 : : Arch $52^{''},\ 44^{'''},\ 03^{vi},\ 45^{v}$: Arch $^{'}$: : S, $52^{''}.\ 44^{'''}.\ 03^{iv}.\ 45^{v}$: S $1^{'}$.

And thus by a continued Biſſection of Arches, the Sine of one Minute being had (by the 3) is found the Sine of two Minutes, then (by the 4) the Sine of three Minutes, and ſo to 30 *deg.* then (by the 5) to 60 *deg.* and (by the 6) the reſt to 90 *deg.*

Having found the Sines, the Tangents, and Secants may be found by the following Proportions.

$$\left.\begin{array}{l} FE.\ FB :: EA.\ BM \\ FE.\ FB :: FA.\ FM \\ BM,\ BF :: FG.\ GI \end{array}\right\}\text{that is}\left\{\begin{array}{l} \Sigma.\ R :: S.\ T. \\ \Sigma.\ R :: R,\ ſ \\ T.\ R :: R,\ \tau. \end{array}\right.$$

BM . MF :: FG : FI . that is T . ſ :: R . σ.

L E M. I.

$\frac{1}{2}V \times R = S\,q\,\frac{1}{2}$ Arch $= \frac{1}{2}S$ Arch $\times$ T. $\frac{1}{2}$ Arch.

1, EB. BN :: AB. FB :: ($\frac{1}{2}$ AB.) BN. $\frac{1}{2}$ FB. *i. e.* V. S. $\frac{1}{2}$ Arch :: S. $\frac{1}{2}$ Arch $\frac{1}{2}$ R. therefore $\frac{1}{2}V R = S\,q\,\frac{1}{2}$ Arch.

2. $\frac{1}{2}$ AE. BN :: AE. AB :: BN. Bλ. therefore $\frac{1}{2}$ AE $\times$ B$\lambda = S\,q\,\frac{1}{2}$ Arch $= \frac{1}{2}S$ Arch $\times$ T $\frac{1}{2}$ Arch.

L E M. II.

$\frac{1}{2}R\,v = \Sigma\,q\,\frac{1}{2}$ Arch $= \frac{1}{2}S$ Arch $\times\,\tau\,\frac{1}{2}$ Arch.

1. $\frac{1}{2}$ ED. FN :: (ED. AD :: AE. AB ::) FN. FB, therefore $\frac{1}{2}$ ED $\times$ FB $=$ FN q, that is $\frac{1}{2}R\,v = \Sigma\,q\,\frac{1}{2}$ Arch.

2. FG. Gz :: AE. ED, therefore $\frac{1}{2}$ AE $\times$ Gz $= \frac{1}{2}$ ED $\times$ FG $=$ FN $q. = \frac{1}{2}S$ Arch $\times\,\tau\,\frac{1}{2}$ Arch.

L E M. III.

The Tangents of two Arches A *and* B, *are recipr*[illegible]*lly proportional to their Co-tangents.*

for T, A. R :: R. τ, A.
and T, B. R :: R. τ, B

---

therefore T, A. $\times\,\tau$ A $=$ T, B $\times\,\tau$ B
therefore T, A. T, B :: τ B τ A.

LEM. IV.

The Co-ſines of two Arches A *and* B, *are reciprocally proportional to their Secants.*

Σ A . R : : R . ſ A
Σ B . R : : R . ſ B.

therefore Σ A × ſ A = Σ B × ſ B
therefore Σ A . Σ B : : ſ B . ſ A.

Plane and Spherical Trigonometry are ſolved by the help of four Fundamental Propoſitions call'd Axioms.

AXIOM I.

For Plane Trigonometry.

In a Right-argled Triangle, if one Leg of the Right-angle, be made the Radius of a Circle, then ſhall the other Leg be the Tangent of the Angle oppoſite to it, and the Hypothenuſe (or ſide oppoſite to the Right-angle) its Secant.

But if the Hypothenuſe be made the Radius of a Circle, then will the Legs (or Sides including the Right-angle) be the Sines of the Angles oppoſite, which will be evident by the Figure.

Upon this Axiom depends the Solution of the 7 Caſes of Right-angled Triangles.

And in the following Proportions, I ſuppoſe that two Lines eſtimated in parts of any Meaſure.

Plate II, *Fig* I. For *Example*, in parts of the Table, are Proportional to themſelves reckon'd according to any other Meaſure; ſo A B reckon'd as Radius, of parts 100000, is to B C the Tangent of the Angle A 30 *deg.* of Tabular parts 57735; ſo the ſame AB of 213 Feet, to BC 123 Feet almoſt.

Note, That the three Angles of a Triangle make 2 ∟ or 180 *deg.* and therefore in a Right-angled Triangle, one of the acute Angles is the Complement of the other to 90 *deg.* conſequently one being given, the other is known; and in an Oblique angled Triangle, two Angles being given, the third is alſo known as being the Suppliment or Reſidue of the Sum of the other two, to 180 *deg.*

For the more eaſie making the Proportions for the Solutions of Right-angled Triangles.

Obſerve, That as different Sides are made Radius ſo the other Sides acquire different Names: Which Names are either Sines, Tangents or Secants, And are to be taken out of your Table.

To find a Side, any Side may be made Radius.

Then ſay,

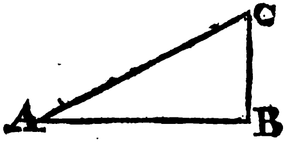
As the Name of the Side given,
Is to the Name of the Side requir'd.
So is the Side given,
To the Side requir'd.
But to find an Angle, one of the given Sides muſt be made Radius.

Then,

As the Side made Radius,
Is to the other Side:
So is the name of the firſt Side,
(Which is Rad.) to the Name of the Second Side.

Which fourth Proportional muſt be found amongſt the Sines or Tangents, &c. to be Determin'd by the Side made Radius and againſt it is the Angle requir'd.

The Proportions for the Solution of the 7 Caſes of Plane Right-angled Triangles,



Given	Req.	Propoſitions.	Rad.	Caſe
AB A & C	BC	Σ A. S, A : : AB. BC R. T, A : : AB. BC τ A. R : : AB. BC	AC AB BC	1
AB A & C	AC	Σ A. R : : AB. AC R. ſ, A : : AB. AC τ A. σ A : : AB. AC	AC AB BC	2
AB BC	A & C	A B. BC : : R. T, A Complement is C. B C. AB : : R. T, C Complement is A.	AB BC	3
AB BC	AC	A B. BC : : R. T, A then, Σ A. R : : AB. AC or √: AB*q* + BC*q* = AC	AB AC	4
AB AC	A & C	A C. AB : : R. Σ A A B. AC : : R. ſ. A	AC AB	5
AB AC	BC	A C. AB : : R. Σ A then, R. T, A : : AB. BC or √: AC*q* — AB*q* = BC	AC AB	6
AC A C	AB	R. Σ, A : : AC. AB ſ, A. R : : AC. AB σ A. τ A : : AC. AB	AC AB BC	7

EXAMPLES.

Caſe I. given AB. 56 *parts*
∠,A. 30 *deg.* } Req. BC.
∠ C ~~∠,B.~~ 60 *deg.*

1. *Geometrically with Scale and Compaſſes.*

From a Scale of equal Parts lay down the Line AB = 56, Erect the Perpendicular thereto BC, at the Point A make the Angle BAC = 30 *deg.* produce the Line AC till it meet the Perpendicular BC in C, and the Triangle is determin'd, and the Line BC being Meaſur'd by the ſame Scale by which you Plotted, AB will give its length.

Plate II.
Fig. II.

2. *By the Table of Natural Sines, Tangents, &c.*

R. T, *A* : : AB. BC.

(*i.e.*) 10000000. 5773503 : : 56.
56

34641018
28867515

32.3316168 that is 32.3316, &c. = BC.

which is found by Multiplying the 2d and 3d Terms into one another, and dividing the Product by the firſt.

To avoid Prolixity I ſhall give no more Solutions by the Table of Natural Sines, &c.

3. *By the Canon of Artificial Sines,* &c. *and Logarithm Numbers*

RULE.

Add the Logarithms of the 2d and 3d Term together, from that Sum Subtract the Logarithm of the firſt Term, the remainder ſhall be the Logarithm of the 4th Term required.

By LOGARITHMS.

T. 30 *deg*. 9.7614394
56. 1.7481880

Sum 11.5096274
Subt. Rad. 10.0000000

Rem. 1.5096274 the Logarithm of 32.33 = BC

Caſe 2. given AB = 56 *parts* }
∠, A. 30 *deg*. } To find AC.
∠C ~~∠, B.~~ 60 *deg*. }

1. *Geometrically.*

Plate II. *Fig.* III. The Triangle being Conſtructed as before, Meaſure AC by the Scale you Plotted AB by, and you have its Length.

2. *By Logarithms.*

≨ A . R : : AB. AC

R. 10.0000000
AB. 1.7481880

Sum. 11.7481880
Subt. ≨ A. 9.9353204

Rem. 1.8128676 the Logarithm of 64.99 = AC

Caſe 3. given AB = 60 }
BC = 35 } parts, to find ∠∠, A&C.

1. *Geometrically.*

Draw the Line AB, and put 60 parts from A to B upon the Point B, let there be the Perpendicular Erected BC, lay off 35 parts from B to C, then draw the Line AC, and Meaſure the Angles A and C by the Line of Chords

Plate II. *Fig.* IV.

2. *By the Logarithms.*

AB. BC : : R. T, A.

BC. 1.5440680
R. 10.0000000

Sum. 11.5440680
Subt. AB. 1.7781512

Rem. 9.7659168 which is the Logarithm Tangent of 30 *deg.* 15 *min.* the Angle; the Complement of which 59.45 is the Angle C.

Caſe 4. given $\left.\begin{array}{l}AB=60\\BC=35\end{array}\right\}$ parts to find AC

1. *Geomtrically.*

The Triangle being Conſtructed, as in the former Caſe, Meaſure AC by your Scale of equal parts and 'tis done.

Plate II.
Fig. V.

2. *By Logarithms.*

Find by the Laſt Caſe the Angle A.
Then by the ſecond Caſe find AC.

Caſe 5. given $\left.\begin{array}{l}AB=80\\AC=100\end{array}\right\}$ parts, to find A and C.

1. *Geometrically.*

Draw the Line AB putting thereon 80 parts from A to B, upon the Point B to the Line A B erect the Perpendicular BC, open your Compaſſes, to the diſtance of 100 equal parts, and putting one Foot in A, with the other croſs the Line BC in C, then draw AC, and Meaſure the Angles A and C by your Line of Chords.

Plate II.
Fig. VI.

2. *By Logarithms.*

AC. AB: :R. Σ A

AB. 1.9030900
R. 10.0000000

Sum 11.9030900
Subt. AC. 2.0000000

Rem. 9.9030900 the Logarithm
Co-ſine of 36 *deg.* 52 *min.*

Caſe 6. given A B=80 } parts, To find B C.
A C=100 }

1. *Geometrically.*

Plate II. The conſtruction being the ſame as in the
Fig. VII. laſt, therefore; Meaſure B C *by your* Seale of equal parts.

2. *By Logarithms.*

By the laſt Caſe find the Angle A:
Then by the 1. Caſe having the Angle A and AB, Given, find B C.

Caſe 7. given A C=50 *parts* }
∠ A=35 *deg.* } To find AB.
∠ C=55 *deg.* }

1. *Geometrically.*

Draw the Indeterminate Right-line A B, at any Point A draw the Right-line A C, making the
Plate II. Angle BAC=35 *deg.* put 50 *parts* from A
Fig. VIII. to C, from C let fall the perpendicular C B to the Line AB, cutting A B in B, AB Meaſur'd By *your* Scale of equal Parts anſwers the Queſtion.

2. *By Logarithms.*

R. ΣA : : AC. AB

ΣA. 9.9133645
AC. 1.6989700

Sum. 11.6123345
Subt. R. 10.0000000

Rem. 1.6123345 the Logarithm of 40.95 parts. =AB.

Note, all theſe Caſes may eaſily be workt by the Gunters Scale, whereon is a Line of Logarithm Numbers, Logarithm Sines, and Logarithm-Tangents.

RULE.

Extend your Compaſſes on this Rule from the firſt Term to the ſecond Term in the Proportion, and the ſame extent will reach from the third Term to the fourth Term requir'd.

Only *obſerve* that if your ſecond Term be $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than the firſt, the fourth Term muſt be $\left\{\begin{matrix}\text{greater}\\ \text{leſs}\end{matrix}\right\}$ than the third, and therefore when your Compaſſes are placed on the third Term, extend them that way the Number muſt be greater or leſs as this Rule directs.

Example in the 1. Caſe

Where the Proportion is R. T,A : : AB. BC.

Extend your Compaſſes from the Radius of the Tangents (which is the Tangent of 45 *deg.*) to the Tangent of the Angle A, &c. 30 *deg.* the ſame Extent of the Compaſſes will reach from the third Term AB=56 *parts*, to 32. 33 *parts*=BC *required.*

Example

Example in the 2. Caſe.

The Proportion is $\gtrless$ A. R :: A B. A C.

Extend your Compaſſes from the Co-ſine of A, which is the Sine of 60 *deg.* to the Radius of the Sines (which is the Sine of 90 *deg.*) the ſame Extent will reach from the third Term A B 56 *parts*, to the fourth Term A C = 54. 99 *parts.*

After the ſame way you may work them all by your Gunter's Scale.

AXIOM II.

Of Oblique Plane Triangles.

In any Triangle, the Sides are Proportional to the Sines of the Angles oppoſite.

DEMONSTRATION.

Produce the leſſer Side A B to F, making A F = B C, let fall the Perpendiculars F E, B D upon the Side A C produced if need be, then will F E be the Sine of the Angle A, and B D the Sine of the Angle C to the Radius B C = AF. Now the Triangles A B D, A F E are Similar; therefore AF (BC) AB :: F E. B D that is :: S, A. S. C, Q. E. D.

Plate III. *Fig.* I. and II.

Otherwiſe thus, A B. R :: B D. S, A
and B C. R :: B D. S, C

therefore A B × S, A = B C × S, C
therefore A B B C :: S, C. S, A Q. E. D.

AXIOM III.

The Sum of the Legs of an Angle is to their difference, as the Tangent of ½ the Sum of the Angles oppoſite to thoſe Legs, to the Tangent of half their Difference.

D. Ex

Plate: 3.

Fig: I.

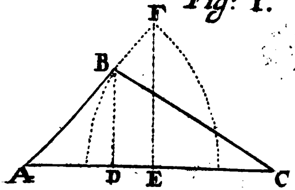


Fig. II.

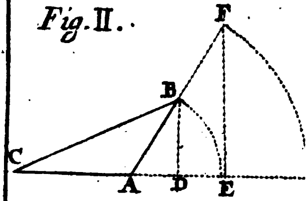


Fig: III.

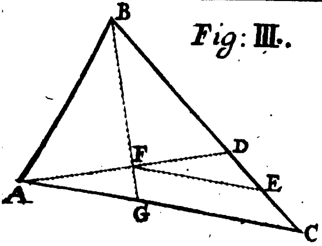


Fig. IIII.

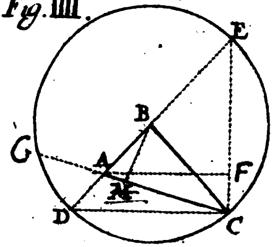


Fig. V.

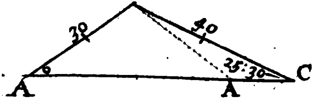


Fig. VI.

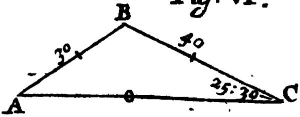


Fig: VII.

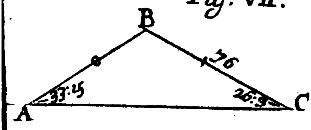


Fig: VIII.

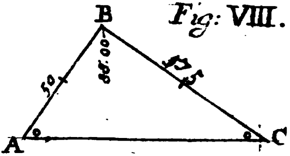


Fig: IX.

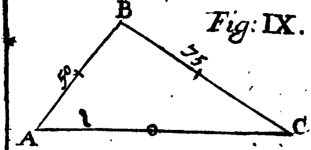


Fig: X.

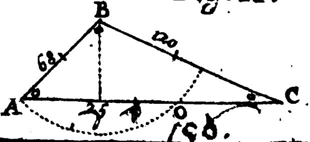
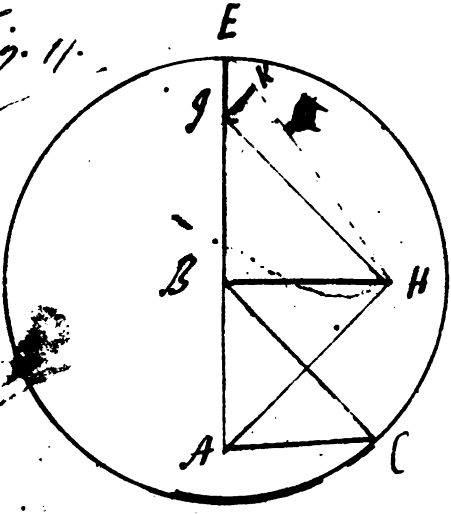
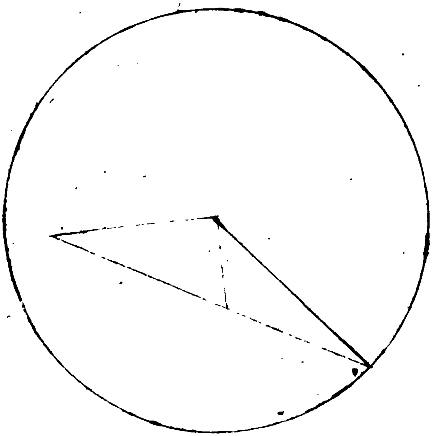


Fig. 11.





DEMONSTRATION.

In the Triangle A B C, make B D = B A, and draw the Line A D, let fall the Perpendicular B F upon the Line A D, which will biſſect A D in F, draw F E Parallel to A C, which will biſſect D C, the difference of the Legs in E, the Angle B A F is ½ the Sum of the Angles B A C and C, and F A G ½ their difference, now if A F be Radius, B F will be the Tangent of ½ the Sum, & F G of the ½ Difference. Now the Triangle B G C is cut Proportionably by the Line F E.

Plate III.
Fig. III.

therefore B E. E C : : B F. F G.
or 2 B E. 2 E C : : B F. F G.

That is,

As the Sum of the 2 Sides to their difference,
So is the Tangent of ½ the Sum of the Angles oppoſite to thoſe Sides,
To the Tangent of ½ their difference,
Q. E. D.

Or thus,

Upon B with the diſtance B C deſcribe the Circle B C E D, Produce A B to E and D, and C A to G, draw D C, and A F Parallel to D C, draw likewiſe C E.

'Tis plain that A E is the Sum of the 2 Sides A B, B C, and A D, their difference, the Angle B A F is ½ the Sum of the Angles oppoſite to thoſe Sides, and the Angle F A C ½ their difference.

Plate III.
Fig. IV.

Now the Angle D C E in a Semi-circle is Right = A F C, and therefore if A F be Radius, F E will be the Tangent of the ½ Sum of the Angles, and F G the Tangent of the ½ difference.

Now the Triangle E D C is cut Proportionably by the Line A F. Therefore A E. AD : : E F. F C. Q. E. D.

PROB.

If the Angle ABC be given, and the Logarithms of the Legs AB, CB: and it be required to find the two remaining Angles BAC, and BCA; which is of frequent uſe in *ASTRONOMY.*

RULE.

As the leſſer Leg AB to the greater BC :: ſo is Radius to the Tangent of an Arch; from which take away 45 degrees.

Then as Rad. to the Tang. of the remaining Arch.
:: So is the Tang. of ½ the Sum of the Angles
A and C, to the Tangent of ½ their difference.

DEMONSTRATION.

Plate III. ~~Fig. IV.~~ Fig. II.

Draw BH perpendicular to AB = AB = BI and IK perpendicular to HI.

Then BH. BE :: R. T, Angle BHE, whence taking away the Angle BHI = 45 *deg.* there willl remain the Angle IHK.

AE: GE And R. T, IHK :: HI. IK :: AH (= HI) IK :: AEI. E :: (by this 3*d Axiom*) T, ½ Z∟∟ op. T. ½ X∟∟ op. Q. E. D.

AXIOM IV.

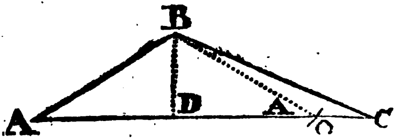
Having drawn a Perpendicular BH from an Angle to its oppoſite Baſe, the Baſe AC ſhall be to the Sum of the other two ſides AE, as their difference DA is to the difference of the Segments of the Baſe GA.

DEMONSTRATION.

For by the 35 Propoſition of the third of *Euclid*, the Rectangle under AD, AE is equal to the Rectangle under GA and AC.

Therefore AC. AE :: DA. GA. Q. E. D.

The Proportions for the Solutions of the 6 Caſes in Oblique Plane Triangles.



Given	Req	Proportions.	Axi.	Caſe.	
AB BC & C	A	AB. BC : : S, C. S, A	2	1	3.K
AB BC & C	AC	AB. BC : : S, C. S, A Hence by Subſtraction the Angle B will be known. S, A. S, B : : BC. AC	2	2	3.K
A C BC	AB	S, A. S, C : : BC. AB	2	3	1.K
B AB BC	A & C	BC + AB. BC — AB : : T $\frac{1}{2}$ Z *LL* oppoſite. T $\frac{1}{2}$ X *LL*, op. then, $\frac{1}{2}$Z + $\frac{1}{2}$X = greater *L*, A $\frac{1}{2}$Z — $\frac{1}{2}$X = leſſer *L*, C	3	4	4.K
B AB BC	AC	1. find the Angles by the laſt then S, C. S, B : : AB. AC	3	5	4K
AB BC AC	A B C	AC. BC + BA : : BC — BA. OC then $\frac{1}{2}$ AC + $\frac{1}{2}$OC = DC $\frac{1}{2}$AC — $\frac{1}{2}$ OC = DA then AB. AD : : R. Σ A. & CB. DC : : R. $\underset{D}{\Sigma}$ C.	4	6	5.K

Examples in the Caſes of Oblique Triangles.

Caſe I. given $\left.\begin{array}{l}AB=30\\BC=40\\\angle C=25\,d.\,30'\end{array}\right\}$ To find the Angle A

1. *Geometrically.*

Plate III. *Fig.* V.

Draw the Line C A produced as far as neceſſary: At the Point C make the Angle A C *B* = 25 *deg.* 30 *min.* ſet off 40 parts from C to *B*, take 30 parts in your Compaſſes from the ſame Scale, and putting one point in *B*, croſs the Line CA, with the other, which it will do in two points A and A. Draw the Lines B A, B A; then the Angles C A B, C A B, being meaſured by a Line of Chords, will give the Anſwer. And here 'tis manifeſt, that the Angle A, in the one Caſe is Obtuſe, in the other Acute.

But here Note, That if the given Angle be Obtuſe, the Angle required is Acute. But if Acute, and oppoſite to the $\left\{\begin{array}{l}\text{greater}\\\text{leſſer}\end{array}\right\}$ ſide the Angle required is $\left\{\begin{array}{l}\text{Acute,}\\\text{Doubtful}\end{array}\right\}$ and whether Acute or Obtuſe muſt be determin'd before the Operations.

2. *To do this by the Canon of Artificial or Logarithmick Sines.*

Obſerve, That it has been demonſtrated in the *ſecond* Axiom, that the Sides have ſuch Proportion one to another, as the Sines of the Angles oppoſite.

Therefore; *Firſt*,

As the Sine of any one Angle,
Is to the Sine of any other Angle;
So is the Side oppoſite to the firſt Angle;
To the Side oppoſite to the ſecond Angle.

Secondly,

As any one Side,
Is to another Side;
So is the Sine of the Angle oppoſite to the firſt Side,
To the Sine of the Angle oppoſite to the ſecond Side.

Therefore, If you would find a Side, begin your Proportion with the Sine of an Angle; and if you would find an Angle, begin with a Side.

AB. BC :: S,C. S,A.

BC. 1.6020600
S,C. 9.6339844

Sum. 11.2360444
A B. ſubſt. 1.4771212

9.7589232

The Sine of A $\left\{\begin{matrix} 35^{d.} & 2' \\ 144 & 58 \end{matrix}\right\}$ if $\left\{\begin{matrix} \text{Acute,} \\ \text{Obtuſe.} \end{matrix}\right.$

Caſe 2. given $\left.\begin{matrix} AB=30 \\ BC=40 \\ C=25\,d.\ 30' \end{matrix}\right\}$ To find A C.

1. Geometrically.

The Conſtruction is already ſhew'd; Therefore, meaſure A C, and you have

Plate III.
Fig. VI.

what is deſir'd, that is the greater CA, if A be acute, the leſſer if the contrary.

2. *By the Canon of Logarithm Sines, &c.*

Find the Angle A by the laſt Caſe: For Inſtance A was found,

$$\left.\begin{array}{l}35\ d.\ 2'\\144:\ 58\end{array}\right\}\text{if}\left\{\begin{array}{l}\text{Acute,}\\\text{Obtuſe.}\end{array}\right\}\text{and conſequently ABC=}\left\{\begin{array}{l}119.\ 28\\9.\ 32\end{array}\right.$$

S, A. S, B : : B C. A C.

S,B.	9.9398396	
BC.	1.6020600	
Sum.	11.5418996	
Subſt. S, A.	9.7589519	
Rem.	1.7829477	60, 66=A C if the Angle A be Acute.
S, B.	9.2191164	
B C.	1,6020600	
Sum.	11.8211764	
Subſt. S, A.	9.7589519	
	1.0622245	11, 54=AC if A be Obtuſe.

$$\text{Caſe 3, given}\left.\begin{array}{l}\text{BC}=76\\\text{A}=33,\ 15'.\\\text{C}=26\ \ 3'\end{array}\right\}\text{To find A B.}$$

1. *Geometrically.*

Plate III. *Fig.* VII.

The Angles A and C being known, the Angle B muſt neceſſarily be known, *viz.* 120*d.* 42'. Draw the Line C B, and thereon put 76 *pts.* from your Scale from C to B, at the Point C make the Angle BCA= 26. 3' and the Angle C B A = 120. 42. the lines CA and BA will concur in A, and the Triangle is determin'd, then if you meaſure A B by your Scale, 'tis done.

2. *By Logarithms.*

S, A. S, C : : BC. AB.

S, C. 9.6426182
B C. 1.8808136

Sum. 11.5224318
Subſt, S, A. 9.7390129

Rem. 1.7844189 the Logarithm of AB. 60.87 *pts.*

Caſe 4. given BC = 75
BA = 50
B = 88 *d.* 00′
} To find the Angles A and C.

1. *Geometrically.*

Draw the Line B C and from your Scale put off 75 from B to C, then draw the Line B A making the Angle C B A = 88*d.* 00′ and ſet off 50 *pts.* from B to A, and draw the Line A C and the Triangle is determin'd. Then the Angles A and C may be meaſur'd by a Line of Chords.

Plate III. *Fig.* VIII.

2. *By Logarithms.*

BC+AB. BC—AB : : T $\frac{1}{2}$ Z ∠ ∠ op. T $\frac{1}{2}$ X ∠ ∠'s Op.
That is,
125. 25 : : T 46*d.* T $\frac{1}{2}$X ∠ ∠'s Op.

25. 1.3979400
T. 46*d.* 10.0151628

Sum. 11.4131028
Subſt. Log. 125. 2.0969100

Rem. 9.3161928 which is the Log. Tang. of $\frac{1}{2}$ the diff. of A and C = 11*d.* 42′

Now $\frac{1}{2}$ the Sum of A & B 46 *d.* $\left\{\begin{array}{l} +\frac{1}{2}\text{ the diff. } 11.42' \\ - \end{array}\right\} = \left\{\begin{array}{l} 57d.\ 42'\text{ the Ang. A} \\ 34d.\ 18'\text{ the Ang. C.} \end{array}\right.$

This hath been prov'd in the third Axiom.

Plate III. Caſe the 5. Given, the ſame as before:
Fig. IX. To find the ſide AC.

The Angles A and C being found as in the laſt Caſe.

A = 57 *d.* 42′
C = 34 18

By Logarithms.

S, C. S, B : : A B. A C.
S, B. 9.9997354
A B. 1.6989700

Sum. 11.6987054
Subſt. S, C. 9.7509140

1.9477914 the Logarithm of 88. 67 *pts.* = A C.

Caſe

$$\text{Caſe 6. given } \left.\begin{array}{l} AB = 68 \\ BC = 120 \\ AC = 156 \end{array}\right\} \text{To find the Angles A, C, B.}$$

1. *Geometrically.*

With the 3 Lines AC, CB, BA make the Triangle ABC, then by the help of a Line of Chords, meaſure the Angles requir'd. *Plate* III. *Fig.* X.

2. *By Logarithms.*

A C. B C + B A :: B C — B A. O C the difference of the Segments of the Baſe.

That is,

156. 188 :: 52. OC.
188 2.2741578
52 1.7160033

Sum 3.9901611
Subſt. 156. 2.1931246

Rem. 1.7970365 = 62.67 = O C the difference of the Segments.

$$\text{Then } \tfrac{1}{2}AC \pm \tfrac{1}{2}OC \begin{array}{l} = DC = 109.33 \\ = DA = 46.67 \end{array}$$

Now the Oblique Triangle A B C is by the Perpendicular BD divided into two Right-angled Triangles B D C, B D A, in each of which the Hypothenuſe and Baſe is given: Therefore by the 5 Caſe.

$$\left.\begin{array}{l} AB.\ AD :: R.\ \Sigma\, A \\ BC.\ DC :: R.\ \Sigma\, C \end{array}\right\} \text{by the 5 Caſe.}$$

The Operation of which I ſhall leave to the Learner to avoid Prolixity.

This

This laſt Caſe may be Solved by four other Theorems.

For the Analyſis and Demonſtration whereof; Suppoſe $m, n.$ the Legs of the Angle required b its Baſe $a, n \backsim a$ the Segments of the Baſe made by p, a Perpendicular let fall from another Angle, whether it ſall within or without the Triangle.

Plate IV. *Fig.* I. and II.

$$\begin{aligned} z &= m + n \\ x &= m - n \\ \text{ʒ} &= \frac{m + n + b}{2} \end{aligned}$$

Then $mm - aa = pp = bb - nn - aa + 2na.$
theref. $a = \dfrac{mm + nn - bb.}{2n}$

therefore $R.\Sigma :: m.\dfrac{mm + nn - bb}{2n} :: 2mn.\ mm +$
$nn - bb$ therefore $2mn.\left(2mn \underset{-}{+} mm \underset{-}{+} nn \underset{+}{-} bb\right) \underset{=}{\ }$
$zz - bb$
$bb - xx$
compounding and dividing $:: R.R \underset{-}{+} \Sigma \doteq \dfrac{v}{V}$ by def. 9th

$$:: \tfrac{1}{2} Rq.\ \left.\begin{matrix} \tfrac{1}{2} Rv \\ \tfrac{1}{2} RV \end{matrix}\right\} \doteq \left\{\begin{matrix} \Sigma q \tfrac{1}{2}\text{ Ang.} \\ Sq \tfrac{1}{2}\text{ Ang.} \end{matrix}\right\} \begin{matrix} \text{by the 2d Lemma.} \\ \text{by the 1ſt Lemma.} \end{matrix}$$

$$\text{Therefore, } 4mn.Rq :: \left\{\begin{matrix} zz - bb \\ bb - xx \end{matrix}\right\} \text{ to } \left\{\begin{matrix} \Sigma q \tfrac{1}{2}\text{ Angle.} \\ Sq \tfrac{1}{2}\text{ Angle.} \end{matrix}\right.$$

and $zz - bb.\ bb. - xx :: \Sigma q \tfrac{1}{2}$ Angle, $Sq \tfrac{1}{2}$ Angle $::$ $Rq.Tq \tfrac{1}{2}$ Angle $:: \tau q \tfrac{1}{2}$ Angle $Rq.$ this will be Evident by the 3d Lemma, and becauſe $\Sigma.S :: R.T.$

Therefore,

$$1.\ 4mn.\overline{z+b}\times\overline{z-b} :: Rq.\Sigma q.\tfrac{1}{2}\ \text{Angle};$$

or dividing the 1ſt and 2d Term by 4.

$$mn.z\times\overline{z-b} :: Rq.\Sigma q\ \tfrac{1}{2}\ \text{Angle}.$$

In Words thus,

As the Rectangle of the Sides including the Angle required.
Is to the Rectangle of ½ the Sum of the 3 Sides Multiplied by ½ the Sum of the 3 Sides made leſs by the Baſe.
So is the Square of the Radius,
To the Square of the Co-ſine of ½ the Angle required.

$$2.\ 4mn.\overline{b+x}\times\overline{b-x} :: Rq.Sq.\tfrac{1}{2}\ \text{Angle},$$

Or as before,

$$mn.\overline{z-m}\times\overline{z-n} :: Rq.Sq\ \tfrac{1}{2}\ \text{Angle}.$$

In Words thus,

As the Rectangle of the Sides including the Angle required.
Is to the Rectangle of ½ the Sum of the 3 Sides, made leſs by one of the Sides including the Angle, Multiplied by ½ the Sum of the 3 Sides made leſs by the other Side including the Angle requir'd:
So is the Square of the Radius;
To the Square of the Sine of ½ the Angle required.

$$3.\ \overline{z+b} \times \overline{z-b} . \overline{b+x} \times \overline{b-x} :: R\,q . T\,q\ \tfrac{1}{2}\ \text{Angle.}$$

Or,

$$\text{ʒ} \times \overline{\text{ʒ}-b} . \overline{\text{ʒ}-m} \times \text{ʒ}-n :: R\,q . T\,q . \tfrac{1}{2}\ \text{Angle.}$$

In Words thus,

As ½ the Sum of the Sides multiplied by ½ the Sum of the Sides made leſs by the Baſe,

Is to half the Sum of the 3 Sides, made leſs by one of the including Sides, multiplied by ½ the Sum of the 3 Sides made leſs by the other inclining Side :

So is the Square of the Radius.

To the Square of the Tangent of ½ the Angle requir'd.

$$4.\ \overline{b+x} \times \overline{b-x} : \overline{z+b} \times \overline{z-b} :: R\,q . \tau\,q . \tfrac{1}{2}\ \text{Angle.}$$

Or,

$$\overline{\text{ʒ}-m} \times \overline{\text{ʒ}-n} . \text{ʒ} \times \text{ʒ}-b :: R\,q . \tau\,q . \tfrac{1}{2}\ \text{Angle.}$$

In Words thus,

As ½ the Sum of the 3 Sides made leſs by one of the including Sides, Multiplied by ½ the Sum of the three Sides made leſs by the other including Side ;

Is to ½ the Sum of the 3 Sides Multiplied by ½ the Sum of the 3 Sides made leſs by the Baſe.

So is the Square of the Radius,

To the Square of the Co-tangent of ½ the Angle required.

Now

$$\text{Now } zz - xx = 4mn,$$
$$\text{For } \tfrac{1}{2}z + \tfrac{1}{2}x = m$$
$$\tfrac{1}{2}z - \tfrac{1}{2}x = n$$
$$\text{therefore } \tfrac{1}{4}zz - \tfrac{1}{4}xx = mn$$
$$\text{and Conſeq. } zz - xx = 4mn.$$

If therefore you have an Angle given, with its Baſe, and the Sum or Difference of the Legs, you'll have by theſe Theorems the Square of the Sum or Difference, and ſo both Sum and Difference, and conſequently the very Legs.

For Inſtance, in the firſt Caſe if the Angle included between the Sides m and n were given together with z the Sum of m and n, then the 2d, 3d and 4th Terms are abſolutely known, and conſequently the 1 Term $4mn$ is known, if from zz I Subſtract $4mn$ the remainder is xx the Square of the Difference, then

$\tfrac{1}{2}z + \tfrac{1}{2}x$ greater Side,
$\tfrac{1}{2}z - \tfrac{1}{2}x$ leſſer Side.

5. In the Theorem which firſt comes out for the finding the verſed Sine of the Angle required.

$$\textit{Viz.}\ 2mn.\ bb - xx :: R.V$$

Let it be changed into theſe two Proportions,

$$\textit{Viz.}\ 2m.\ b+x :: b-x.\ G = \frac{bb-xx}{2m}$$

$$\text{then. } n.\ G\left(=\frac{bb-xx}{2m}\right) :: R.\ V = R \times \frac{bb-xx}{2mn}$$

Or into theſe Two.

$$b+x.\ 2n :: m.\ H = \frac{2mn}{b+x}$$

$$\text{then } H = \left(\frac{2mn}{b+x}\right).\ b - x.\ R.\ V = R \times \frac{bb-xx}{2mn}$$

Which Proportions may be workt by the *Sector* or *Gunter's* Line.

Spheric Trigonometry.

A *Spheric Triangle*, is that which is contain'd between the Arches of 3 great Circles of the Sphere.

A *Spheric Angle* is the ſame with the Inclination of the Planes of thoſe 2 Circles which conſtitute the Angle.

Affections of Spheric Triangles.

1. When a Circle falls on another Circle, the Sum of the two Angles made thereby is = 2 ∟.

2. When a Circle croſſes a Circle, the Vertical Angles made thereby are mutually Equal.

3. The greater Angle is oppoſite to the greater Side:

4. An Iſoſceles Triangle hath its two Angles at the Baſe mutually Equal; and on the contrary, if a Triangle has 2 Angles equal, it has two Sides equal.

5. Two Triangles mutually Equilateral, are alſo Equiangular one to another.

6. If there be two Triangles, and in each one Angle and the two Sides including reſpectively Equal; or if one Side, and the two Angles adjacent be ſeverally Equal; then the two Triangles are equal; for if laid one upon another they will agree.

7. Two Sides of a Triangle are bigger than one; for the Arch of a great Circle is the ſhorteſt Diſtance between two Points on the Surface of a Sphere as a ſtreight Line is betwixt two Points in a Plane.

Theſe ſeven Properties are common to plane Triangles, and have a like Demonſtration.

8. All great Circles mutually cut each other into two equal Parts, for the common Section of their Planes is a Diameter of the Sphere, and conſequently the two Sections of their Peripheries, are at a Semicircle Diſtance.

Hence it follows that every Side of a Spheric Triangle is leſs than a Semicircle. DB is leſs than the Semicircle DC.

Plate IV. *Fig.* III.

9. The oppoſite Angles at the Sections of two Circles are equal $\angle D = \angle C$, for the ſame Planes make both Angles.

10. In any Spheric Triangle if the Sum of the Legs of an Angle DB, BA be $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than a Semicircle DBC the intern Angle, at the Baſe D, is accordingly $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than the outward oppoſite Angle BAC, and conſequently the Sum of the two internal Angles at the Baſe D, DAB ſhall be $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than two $\lrcorner$.

DEMONSTRATION.

If $DB + BA$ be $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than DC, then BA is $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than BC; therefore the Angle C, or which is equal to it D is $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than BAC, and the $\angle D + \angle DAB$ $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than $(\angle BAC + \angle DAB =)\ 2 \lrcorner$.

11. Coroll. In an Iſoſceles Triangle, if one of the equal Legs is $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than a Quadrant, the Angle at the Baſe is $\left.\begin{matrix}>\\=\\<\end{matrix}\right\}$ than $\lrcorner$.

12. The Sum of the three Sides of a Triangle is leſs than a Circle for $BA < BC + AC$; therefore $DB + DA + BA < DBC + DAC$.

13. If from the Point of an Angle as a Pole, you deſcribe a great Circle, or which is the ſame, if you deſcribe a Circle at the diſtance of 90 Deg. from that Point, the Arc of this Circle intercepted between the Legs of the Angle, is the meaſure of the Angle.

14. The Poles of the ſides of any Triangle G H D, conſtitute another Triangle $n\,x\,m$, which we may call Supplemental to the Triangle G H D; For the Supplements of the $\left\{\begin{matrix}\text{Angles}\\ \text{Sides}\end{matrix}\right\}$ of the $\triangle\ n\,x\,m$ are equal to the $\left\{\begin{matrix}\text{Sides}\\ \text{Angles}\end{matrix}\right\}$ of the Trine G H D.

DEMONSTRATION.

From the Points G H D as Poles, deſcribe on the Globe 3 great Circles, xAy, $RTmn$, $xBnZ$, then m being the Pole of G H, my is = Quadrant = Ax, x or E being the Pole of G D, therefore adding mA on both ſides $mx = Ay =$ Supplement of C A, the meaſure of the $\angle$ H G D.

$Zn =$ Quadrant, (n being the Pole of H D) $= Bx$ and adding Bn on both ſides, $nx = BZ =$ Supplement of B K the meaſure of the Angle H D G.

Plate IV. *Fig.* VII.

$nT =$ Quadrant $= mR$, and Subſtracting mT on both ſides $nm = TR$ the meaſure of the Supplement of $\angle$ G H D. And here obſerve that the Triangle nEm conſtituted between the 3 next Poles, hath its 3 $\left\{\begin{matrix}\text{Sides}\\ \text{Angles}\end{matrix}\right\} = \left\{\begin{matrix}\text{Angles}\\ \text{Sides}\end{matrix}\right\}$ Of the Triangle G H D, ſave that the greateſt ſide nm is the Supplement of the Angle H, and the Angle E of the ſide G D.

15. Any Angle of a Triangle with the difference of the other two is $< 2\lrcorner$ for $xn < xm + mn$. i. e. $2\lrcorner - D < 2\lrcorner - G + 2\lrcorner - H$, and taking away $2\lrcorner$ on each ſide. $-D < 2\lrcorner - G - H$, and adding G and H on both ſides $G + H - D < 2\lrcorner$.

Plate IV.
Fig. IV. and VI.

16. 2 Triangles A and B mutually Equiangular, are alſo mutually Equilateral, for becauſe they are Equiangular, their Supplimental Triangles α, β are Equilateral by the 14th, and therefore Equiangular by the 5th; and therefore the propoſed Triangles A and B whoſe ſides are the Supplements of thoſe Angles, are Equilateral by the 14th.

17. Three Angles of every Triangle are $> 2\lrcorner$, and $< 6\lrcorner$ for $nx + xm + nm < 4\lrcorner$ (by the 12th) that is $2\lrcorner - D + 2\lrcorner - G + 2\lrcorner - H < 4\lrcorner$ that is, $6\lrcorner - D - G - H < 4\lrcorner$ and taking away $4\lrcorner$ on each ſide, $2\lrcorner - D - G - H < 0$ and adding D, G, H, on both ſides $2\lrcorner < D + G + H$.

2ly. The Sum of the Intern Angles is leſs than the Sum of the Intern and Extern, both which in all make but $6\lrcorner$.

Or thus, by the Figure to the 14th, the Supplement of nm is the meaſure of the Angle H; E m the meaſure of the Angle G, and E n of D.

But the Supplement of nm added to nm makes but $2\lrcorner$ therefore the Supplement nm added to $Em + En$ is $> 2\lrcorner$, that is $G + D > 2\lrcorner$.

Plate IV.
Fig. V.

Or thus, by the 10th, the Angle B A C is $\left.\begin{matrix} > \\ = \\ < \end{matrix}\right\}$ than the Angle D. In this firſt caſe, make $FAC = D$ and produce A B to E.

Then $DF + AF = 2\lrcorner$, therefore $BF + AF < 2\lrcorner$, therefore $\angle FBE = DBA > FAB$, but $FAD + FDA = 2\lrcorner$; i. e. $FAB + BAD + BDA = 2\lrcorner$, therefore $DBA + BAD + BDA > 2\lrcorner$.

2. If $BAC = D$, then $BAD + BDA = 2\lrcorner$, therefore $BAD + BDA + ABD > 2\lrcorner$.

3. If $BAC < D$, then $BAD + BDA > 2\lrcorner$, therefore $BAD + BDA + ABD > 2\lrcorner$.

Plate IV.
Fig. VIII.

18. Of ſeveral Arcs of great Circles falling from the ſame Point of the Spheres Surface on another Circle, the greateſt is that which paſſes thro' the Pole of the Circle and the next to this,

is greater than that which is farther off. For ſuppoſe P the Pole of the Circle, C ω D and ω the Pole of D P C; then is A D (A P + P B) $>$ A B $>$ A E $>$ A C and the Arc B ω C $>$ B F $>$ B D.

19. A great Circle paſſing thro' the Poles of another great Circle cuts it at Right Angles; and on the contrary, if it cuts it at Right Angles it paſſes thro it's Poles. For the Angle of the Inclination of the two Planes is then right $\angle$ P B D $=$ $\lrcorner$ $=$ P G D $=$ P D B $=$ ω A C.

20. In an Oblique Angled Triangle, if the Angles at the Baſe are like, that is, both Acute, or both Obtuſe, the Perpendicular let fall from the oppoſit Angle falls within the Triangle, and the Quadrantal Arc without: but if they be unlike, the Perpendicular falls without and the Quadrant within, for the Triangle E A F has $\angle\angle$ E, F, Acute and the Perpendicular A C falls within, and the Quadrant A ω without, alſo the Triangle B A G has $\angle\angle$ B, G, Obtuſe, and the Perpendicular A D within, and the Quadrant A ω without.

But the Triangle B A E has the $\angle\angle$ B, E, of different kinds, and the Perpendicular A C without, and the Quadrant A ω within.

By the ſame Figure may the Ambiguities of Right Angled Triangles be Solved.

SOLUTIONS.

1. The Legs of the Right Angle are of the ſame kind with the oppoſite Angles, ſo in the Triangle B D A, becauſe D A $>$ Quadrant D P, the Angle DBA $>$ $\lrcorner$ D B P, and in the Triangle B C A, becauſe A C $<$ Quadrant P C, $\angle$ C B A $<$ $\lrcorner$ C B P.

2. If the Legs (and conſequently the Angles) are of the ſame, or of different kind; the *Hypothenuſe* is accordingly $<$ $>$ Quadrant. So in the Triangles E D A, E C A, the *Hypothenuſe* A E is $<$ Quadrant, but in the Triangle B D A, the *Hypothenuſe* A B is $>$ Quadrant B P.

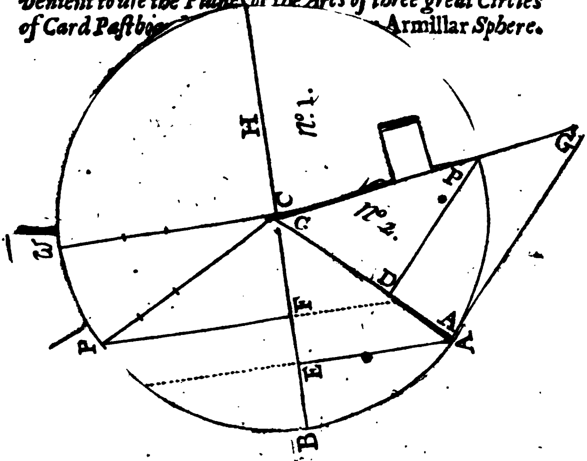
3. *Vice Verſa*, if the *Hypothenuſe* $<$ 90 deg. the Legs are Similar, and the 2 Oblique Angles are Similar; but if

the *Hypothenuſe* be $>$ 90 deg. the Legs are Diſſimilar, and the Oblique Angles Diſſimilar.

4. If the *Hypothenuſe* be $\lessgtr$ 90 Deg. each Leg ſhall be $\left\{\begin{matrix}\text{Similar}\\ \text{Diſſimilar}\end{matrix}\right\}$ to its adjacent Angle, this follows from the 1ſt, and 2d. Sol.

5. And on the contrary, if a Leg be $\left\{\begin{matrix}\text{Similar}\\ \text{Diſſimilar}\end{matrix}\right\}$ to its adjacent Angle, the *Hypothenuſe* will be $\lessgtr$ 90 deg.

For viewing the Sines, Co-Sines, *and other Right Lines of Arcs, which are not viſible in a common Sphere, 'tis convenient to uſe the Planes of the Arcs of three great Circles of Card Paſtbo*[illegible] Armillar *Sphere.*



As ſuppoſe in the preceding Fig. 2 Arcs, B P, B A, and that B P H the Plane of the greater Arc were turn'd round B H, till that a Right Line falling from P perpendicular to the Plane B A H, may fall on ſome Point of the Line C A, ſuppoſe on D; for in that Poſition P A B will be a Spheric Triangle, Right Angled at A, B P the *Hypothenuſe*, B A the Baſe, P A the Perpendicular Arc.

And ſuppoſe P A Num. 2. = P A of the Triangle, and fitted according to its Letters therein; and Draw A E, P F, Perpendicular to B C; ſo A E, P F, P D will be the Sines of the Arcs B A, B P, P A, and their Co-ſines will be E C, F C, D C; theſe things being done or conceived, the two firſt Axioms of *Spheric Trigonometry* will preſently appear, and conſequently the Demonſtration of the 16 Caſes of *Right-Angled Triangles*; to which End let the Arc ϖ α Num. 3. be alſo fitted in the Solid according to its Letters. Then in the 2 *Right-Angled Spheric Triangles* P B A, ϖ B α having the ſame Acute Angle B at the Baſe

AXIOM I.

The Sines of the *Hypothenuſes* are proportional to the Sines of the Perpendiculars.

P F. P D : : ϖ C. ϖ δ.

AXIOM II.

The Sines of the Baſes are proportional to the *Tangents* of the Perpendiculars.

A E. A G : : α C. α γ.

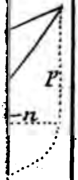
For the two *Right Angle Triangles* F P D, c ϖ δ are Simillar. Alſo, E A G, C α γ are Similar.

For the Solution of the following Caſes I ſuppoſe B P A a *Right Angled Triangle*, and its Sides produced to Quadrants, B N B M, A D; ſuppoſe alſo P E, P F, N G, and E G Quadrants. *Plate* V. *Fig*. I.

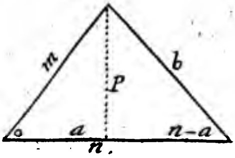
Then is N E = B P, and the Complement of B A = A M = ∠ A D M and F E = ∠ F P E = ∠ B P A, and G D = N M = ∠ B, and the Angles at A, M, N, E, F, right.

Here follows the Table of the Proportions for the Solution of the 16 Caſes of *Right-Angled Spheric Triangles* with the Solution of their Ambiguities.

Examples



2.



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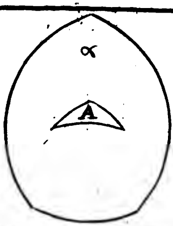

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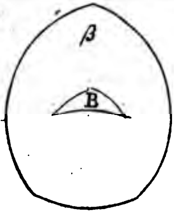

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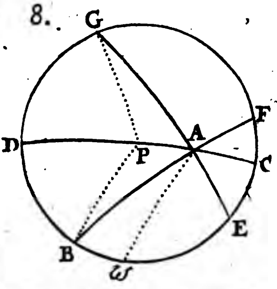

	Given.	Req	Given.	Reqd.	Numb.
4.	BA, PA.	B	*c c.*	*b*	1
6.	BA, PA.	B.	*c c.*	*L*	2
16	BP, P.	B.	*b, L.*	*L*	3
9	BP, P.	P A	*b, L.*	*L*, adj.	4
12	BP, P.	B A	*b, L.*	*c.* op.	5
10	PA, P.	B	*c. L*,adj.	*b*	6
	PA, P.	B	*c. L*, adj.	*L*	7
	B, P.	B P	*LL.*	*b*	15
	B, P.	P A	*LL.*	*c*	16

between page 38, 39.

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Examples in Numbers.

$$\begin{array}{lll} & \quad\text{d}\quad ' & \\ \text{CAſe I. Given BA} = & 30.\ 00 & \multirow{2}{*}{\Big\}\ \text{Req. BP.}} \\ \text{PA} = & 25.\ 00 & \end{array}$$

Plate V,
Fig. I,

PROPORTION,

R. Σ BA : : Σ PA : Σ BP.

Σ BA.	9.9375306
Σ PA.	9.9572757
Sum.	19.8948063
Subſt. R.	10.0000000
Rem.	9.8948063 the Σ of 38.18 = BP

By the Gunter.

Extend your Compaſſes from the Radius in the Line of Artificial Sines, to the Sine of 60 deg. (which is the Complement of BA) the ſame Extent of the Compaſſes will reach from the Sine of 65 (which is the Complement of PA) to 51d 42′ the Complement of 38 deg. 18 min. = BP,

Caſe 2.

Plate V.
Fig. II.

Caſe 2. Given A B=30. 00 } Reqd.B
P A=25. 00 }

PROPORTION.

S, B A. R : : T, P A. T. B.

R.	10.0000000
T. P A.	9.6686725
Sum.	19.6686725
Subſt. S, B A.	9.6989700
Rem.	9.9697025 T, B = 43 deg.

By the 1ſt *Sol.*

By the Gunter.

Extend your Compaſſes from the Sine of B A 30 deg. to the Radius, the Sine of 90 deg. the ſame Extent of the Compaſſes will reach on the Line of Artificial Tangents, from the Tangent of P A 25 deg. to the Tangent of B. 43 deg.

Plate V.
Fig. III.

Caſe 3. Given BP=60d. 00′ } P=50. 00 } Reqd B.

PROPORTION.

R. ≨ BP : : T, P. τ B.

≨ BP 9.6989700
T, P. 10.0761865

Sum 19.7751565
Subſt. R. 10........

Rem. 9.7751565 the τ B=59d. 13′
Sol. the 3d.

By the Gunter.

Obſerve the Method in the former Caſes.

Caſe 4. Given BP=60d. 00′ } P =50. 00 } Required P A.

Plate V.
Fig. IV.

PROPORTION.

R. ≨ P : : T, BP.

≨ P 9.8080675
T, BP 10.2385606

Sum 20.0466281
Sub, R. 10........

Rem. 10.0466281 T, P A=48d. 4′.
Sol. the 4th.

Caſe

d ′
Caſe 5. Given B P = 60. 00 } Reqd. Fig. V.
P = 50. 00 } B A. Plate V.

PROPORTION.

R. S, BP : : S, P. S, BA.

S, B P. 9.9375306
S, P 9.8842540

Sum. 19.8217846
d ′
Subt. R. Remaind. 9.8217846 *S*, B A = 41, 33
Sol. 1ſt.

d ′
Caſe 6. Given P A = 36. 00 } Reqd. B P. *Plate* V.
P = 30. 00 } *Fig.* VI.

PROPORTION.

Σ P. R : : T, P A. T, B P.

R 10.0000000
T, P A 9.8612610

Sum 19.8612610
Subſt. Σ P. 9.9375306
d ′
Rem. 9.9237304 T, B P = 39. 59
Sol. 5.

Caſe 7.

Plate V. d '
Fig. VII. Case 7. Given P A = 36. 00 } Reqd. B.
P = 30. 00 }

PROPORTION

R, ⧖ PA : : S, P. ⧖ B.

⧖ P A 9.9079576
S,P. 9.6989700

Sum 19.6069276 d '
Subst. R. Rem. 9.6069276 ⧖ B 66. 09
Sol. 1ſt.

d '
Plate V. Caſe 8. Given B A = 60. 00 } Reqd. P. A.
Fig. VIII. B = 37. 30 }

PROPORTION.

R, S, B A : : T, B. T, P A.

S, B A 9.9375306
T, B. 9.8849805

Sum 19.8225111 d '
Subſt. R. Rem. 9.8225111 T, P A = 33. 36
Sol. 1ſt.

Caſe 9.

Plate V.
Fig. IX.

Caſe 9. Given $PA = 16^{d}.\ 17'$
$B = 24.\ 40$ } Reqd. B A

PROPORTION.

T, B. R : : T, P A. S, B A.

R. 10.0000000
T, P A 9.4655386

Sum 19.4655386
Sub. T, B 9.6620434

Rem. 9.8034952, S, B A $= 39^{d}.\ 30'$. Or $140^{d}.30'$
This Caſe is Ambiguous.

Plate V.
Fig. X.

Caſe 10. Given $PA = 16^{d}.\ 17'$
$B = 24.\ 40$ } Reqd. B P

PROPORTION.

S, B. R : : S, P A. S, B P.

R. 10.0000000
S, P A. 9.4477586

Sum 19.4477586
Subſt. S, B. 9.6204884

Rem. 9.8272702, S, B P $= 42^{d}.12'$, or $137^{d}.48'$
Ambiguous.

Caſe

d '

Caſe 11. P A = 16. 17 } Reqd. Plate V.
B = 24. 40 } P. Fig. XI.

PROPORTION.

Σ P A. R :: Σ B. S, P.

R. 10.0000000
Σ B. 9.9584450

Sum 19.9584450
Subſt. Σ P A. 9.9822201

d '

Rem. 9.9762249 S, P = 71. 13. If P A falls between N M and B, if not 'tis 108. 47 *Ambiguous.*

d '

Caſe 12. Given P A = 37. 00 } Reqd. P. Plate V.
PB = 46. 00 } Fig. XII.

PROPORTION.

T, B P. R :: T, P A. Σ P.

R. 10.0000000
T, P A. 9.8771144

Sum. 19.8771144
Subſt. T, B P. 10.0151668

d '

Rem. 9.8619516 Σ P = 43. 18.

By the 3d Solution.

Plate V. *Fig.* XIII.

Caſe 13. Given P B = 37° 00′, P B = 46° 00′ } Reqd. B

PROPORTION.

S, B P. R : : S, P A, S, B.

R. 10.0000000
S, P A. 9.7794630

Sum 19.7794630
Subſt. S, B P. 9.8579341

Rem. 9.9215289 S, B = 56° 35′ *Sol.* 1.

Plate V. *Fig.* XIV.

Caſe 14. Given P A = 37° 00′, P B = 46° 00′ } Reqd. B A

PROPORTION.

Σ P A. R : : Σ P B. Σ B A.

R. 10.0000000
Σ P B 9.8417713

Sum 19.8417713
Sub. Σ P A. 9.9023486

Rem. 9.9394227 Σ B A 4 = 29° 34′
By the 2d Solutio

Caſe

d ′
Caſe 15. Given B=46. 00 } Reqd. BP *Plate* V.
P=59. 00 } *Fig.* XV.

PROPORTION.

T, P. R : : τ B. Σ BP.

R. 10.0000000
τ B. 9.9848372

Sum. 19.9848372
Subſt T, P. 10.2212263

d ′
Rem. 9.7636109 Σ BP = 54. 32 *Sol.* 2.

d ′
Plate V. Caſe 16. Given B=46. 00 } Reqd. P A.
Fig. XVI. P=59. 00 }

PROPORTION.

S, P. R : : Σ B. Σ P A.

R. 10.0000000
Σ B. 9.8417713

Sum. 19.4817713
Subſt. S, P. 9.9330656

d ′
Rem. 9.9087057 Σ P A = 35 52

Any Oblique Quadrantal Triangle, Z H P whoſe ſide H Z is a Quadrant, may be reduced to the Right-angled Triangle H N P by producing the other ſide Z P to N, ſo that Z N be Quadrant; (or taking away a Quadrant if Z P had been bigger than a Quadrant) then in the Right-

Plate V. *Fig.* XVII.

angled Triangle H N P, H N = ∠ Z, and PN the Complement of Z P. and ∠ P H N the Complement of ∠ H P.

The Lord Neper from a diligent Obſervation of the *Proportions for Solving the Caſes* before-mention'd, found they might be ſolv'd by one *General Propoſition.*

For the Underſtanding of which, 'tis neceſſary to premiſe that he calls the two *Legs* (*viz.* the *Sides* including the Right Angle) together with the Complements of the other 3 parts *viz.* the *Hypothenuſe* and 2 Oblique Angles) *five Circular Parts*, taking no Notice of the Right Angle as being known; and if the 3 parts which enter the Queſtion (of which 2 are given and 1 requir'd) have no interruption (a Right Angle coming between being accounted none) that part which is between the other two, is call'd the middle part, and the other two *Extremes Conjunct*; but if there be an Interruption, that part which is ſeparated from the other two is call'd the *Middle Part*, and the other 2 *Extremes Oppoſit* or *Disjunct*.

For Inſtance in the Right-angled Triangle B P A.

If		Extr.Conj.		Extr.Disj.
B A	be mid. parts then	Comp. B P A	[illegible]	Comp. P Comp.BP
P A		B A Comp. P		Comp.BP Comp. B
Com. P		Comp.BP P A		Comp. B B A
Com. B C		Comp. B Comp. P		B A P A
Com. B		B A Comp.BP		Comp. P P A.

The

The Lord Neper's *General Propoſition is thus.*

$R \times S$, Middle part $= \square$ Tang. of the Extreme Conjunct;
And $R \times S$, Middle part $= \square$ Coſines of the Extremes Disjunct.

That is, *The Radius multiplied by the Sine of the Middle part is equal to the Rectangle or Product of the Tangents of the Extremes Conjunct, and equal to the Rectangle or Product of the Co-ſines of the Extremes Disjunct.*

Hence, To find the Middle part, when the other two parts given are either $\left\{\begin{array}{l}\text{Extremes Conjunct,}\\ \text{Extremes Disjunct,}\end{array}\right.$

RULE I.

R. T, one Extreme : : T, other Extreme. S, Mid. part
R. Σ one Extreme : : Σ other Extr. S, Middle part.

Here you may obſerve, That when the Middle part is to be found, you muſt begin with the Radius: But when you are to find an Extreme either $\left\{\begin{array}{l}\text{Conjunct,}\\ \text{Disjunct.}\end{array}\right.$

RULE II.

T, one Extr. R : : S, Midd. part, T, other Extr.
Σ, one Extr. R : : S, Midd. part, Σ other Extr.

So that when one Extreme is to be found, the other Extreme muſt be the firſt Term.

Some few Examples to Exerciſe this Rule.

Caſe 1. $\left.\begin{array}{l}\text{B A}=30d\\ \text{P A}=25\end{array}\right\}$ Required B P.

Here the Complement of B P required is the middle part, and the other two parts given are Extremes disjunct; therefore by the 2d part of Rule 1ſt.

R. Σ one Extreme : : Σ other Extreme S mid. part

That is, R. Σ B A : : Σ P A. Σ B P.

the ſame Proportion as before, which being workt in Numbers already, would be needleſs to repeat.

Caſe 2. Where there are the ſame things given to find the Angle B.

Here B A is the middle part, and the Complement of B and P A are Extremes Conjunct.

Therefore, by the 1ſt part of the 2d Rule.

T, one Extream R. : : S, middle part, T, other Extreme that is T, P A. R. : : S, B A. τ B.

Caſe 3. Given $\left.\begin{array}{l}\text{B P}=60\ d.\\ \text{P}=50\end{array}\right\}$ Required B.

Here the Complement of B P is the middle part, and the Complements of B and P are Extremes conjunct.

Then by the 1ſt part of the 2d Rule.

T, one Extreme R. S, middle part, T, other Extreme.

That is, τ P. R : : Σ B P. τ B.

Here though the 2 laſt Canons are not exactly the ſame as thoſe produced in the former Solution of them, derived from the Production of the Sides of the Triangle, yet their Co-incidence with them will appear eaſily from the 3d Lemma.

I preſume 'twill be needleſs to give any farther Inſtances of this General Rule of the Lord *Neper*.

Of the Solution of Oblique Spherical Triangles.

Plate V. *Fig.*XVIII.

OBlique Spherical Triangles may be reduced to two Right-angled Spherical Triangles, by letting fall a Perpendicular, which Perpendicular either divides the Oblique propos'd into two Right, or makes two Right, by adding a Right-angle Triangle to it.

In Oblique Triangles there are 12 Caſes, 10 of which may be ſolved (being thus prepared) by the 2 firſt Axioms, or by Rules deduced from them,

RULE I,

The Co-ſines of the Angles at the Baſe, are proportional to the Sines of the Angles at the Vertex.

Note, That B A, D A are the Baſes of the Right-angled Triangles B P A, D P A.
The Angles B, and D are the Angles at the Baſe; the Angles B P A, D P A are called Angles at the Vertex, B P and D P are called the Sides.

For by the 7th Caſe,

Plate VI. *Fig.* I.

R. Σ P A : : S, B P A. Σ B
R. Σ P A : : S, D P A. Σ D.

Therefore, Σ B. S, B P A : : Σ D. S, D P A.

For Proportions which are the ſame to one third, are the ſame one to another.

RULE II.

The Co-ſines of the Sides are proportional to the Co-ſine of the Baſes.

For by the 1ſt Caſe,
R. Σ PA : : Σ BA. Σ BP. *Plate* VI.
R. Σ PA : : Σ DA. Σ DP. *Fig.* II,
Therefore, Σ BA, Σ BP : : Σ DA, Σ DP.

RULE III.

The Sines of the Baſes are reciprocally proportional to the Tangents of the Angles at the Baſe.

For by the 2d Axiom,

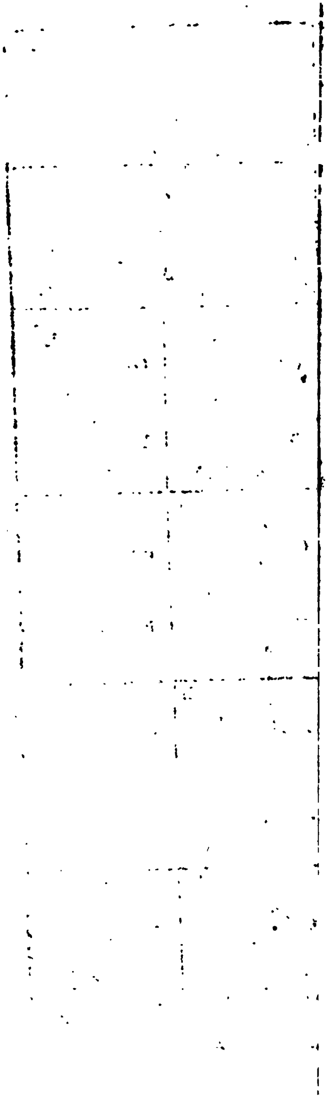
S, BA. R : : T, PA. T, B
S, DA. R : : T, PA. T, D.

Therefore, S, BA × T, B = R × T, PA = S, DA × T, D.

Therefore, S, BA. S, DA : : T, D. T, B.
for Equal Rectangles have their Sides reciprocally Proportional.

RULE IV.

The Tangents of the Sides, are reciprocally Proportional to the Co-ſines of the Angles at the Vertex.



For by the 4th Caſe,

Plate VI. T, B P. R : : T, P A. Σ B P A.
Fig. III. T, D P. R : : T, P A. Σ D P A.

Therefore, T, B P x Σ B P A = R x T, P A = T, D P x Σ D P A.

Therefore, T, B P. T, D P : : Σ D P A. Σ B P A.

AXIOM III.

In any Triangle (whether Right-angled or Oblique) the Sines of the Sides are proportional to the Sines of the Oppoſite Angles.

For by the 1ſt Axiom,

S, B P. R : : S, P A. S, B
S, D P. R : : S, P A. S, D.

Therefore, S, B P x S, B = R x S, P A = S, D P x S, D.

Therefore, S, B P. S, D P : : S, D. S, B.

Obſerve this Rule for the letting fall your pendicular.

Let it be let fall from the End of a given Side, and Oppoſite to a given Angle.

For by ſo doing you have enough given in one of the Right-angled Triangles to Determine any of its unknown parts.

The

The two firſt Caſes are ſolved each by one Operation by the 2d Axiom foregoing, the next eight Caſes are ſolved each by two Operations.

And that you may know what part of the Right-angled Triangle to find in your firſt Operation, *Obſerve to find ſuch a Part, which when found, the other part of the thing required, (which is to be added to, or ſubſtracted from the 1ſt Part found) may come under one of the Four Rules mention'd.*

This will be eaſily underſtood from this General View of the Proportions for the Solution of the firſt 10 Caſes of Oblique Triangles, annexed.

Proportions for the Solving the firſt 10 Caſes of Oblique Triangles.

Given.	Req	Proportions.	Caſe
BP.PD, B	D	S, PD. S, B : : S, B P. S, D. *ambig*	1
B P. B. D.	P D	S, D. S, BP : : S, B. S, PD. *ambig.*	2
BP. PD. B	B D	Firſt, find B A. by the 4th Caſe. R. Σ B : : T, B P. T, B A. And then by the 2d Rule, Σ B P. Σ B A : : Σ D P. Σ D A. Then B A ± D A = B D, according as the Perpendicular falls within or without the Triangle, which is doubtful, unleſs the kind of the Angle D were known.	3
BP.PD.B	P	Firſt, find B P A. by the 3d Caſe Σ B P. R : : τ B. T, B P A. Then by the 4th Rule. T, DP. T, BP : : Σ BPA. Σ DPA. Then, B P A ± D P A = P B D. and here the falling of P A within or without is doubrful, unleſs the kind of the Angle D be known.	4
B P. B. D	P	Firſt find B P A by the 3d Caſe. Σ B P. R : : τ B. T, B P A. Then by the 1ſt Rule. Σ B. S, B P A : : Σ D. S, D P A. Then B P A ± D P A = B P D, that { B and D are alike, is if { B and D are unlike.	5

Proportions for the Solving the firſt 10 Caſes of Oblique Triangles.

Given.	Req.	Proportions.	Caſe
B P. B.D.	B D.	First, find B A. by the 4th Caſe. Σ B. R : : τ B P. τ B A. Then by the 3d Rule T, D. T, B : : S, B A. S, D A. then B A ± D A = B D if $\left\{\begin{matrix}\text{B and D}\\ \text{B and D}\end{matrix}\right\}$ are $\left\{\begin{matrix}\text{like}\\ \text{unlike.}\end{matrix}\right.$	6
B P. B. P.	D.	Firſt, find B P A. by the 3d Caſe, Σ B P. R : : τ B. T, B P A. then by the 1ſt Rule S, B P A. S, D P A : : Σ B. Σ D. if B P A is greater than B P D & alſo B $\left\{\begin{matrix}\text{Acute}\\ \text{Obtuſe}\end{matrix}\right\}$ D is $\left\{\begin{matrix}\text{Obtuſe}\\ \text{Acute.}\end{matrix}\right\}$ but if B P A is < B P D, and B $\left\{\begin{matrix}\text{Acute}\\ \text{Obtuſe.}\end{matrix}\right\}$ D is $\left\{\begin{matrix}\text{Acute}\\ \text{Obtuſe}\end{matrix}\right.$	7
B. P. B P.	P D	Firſt, find B P A. by the 3d Caſe. Σ B P. R : : τ B. T, B P A. then by the 4th Rule Σ D P A. Σ B P A : : T, BP. T, DP if D P A is $\left\{\begin{matrix}\text{like}\\ \text{unlike}\end{matrix}\right\}$ B (or P A) DP is $\left\{\begin{matrix}\text{leſs}\\ \text{greater}\end{matrix}\right\}$ than Quadrant.	8

B P.

Proportions for the Solving the firſt 10 Caſes of Oblique Triangles.

Given	Req.	Proportions.	Caſe
BP.BD.B.	D P	Firſt, find B A. by the 4th Caſe, Σ B. R : : τ B P. τ B A. Then by the 2d Rule Σ B A. Σ B P : : Σ D A. Σ D P. If DA is $\left\{\begin{matrix}\text{like}\\ \text{unlike}\end{matrix}\right\}$ (P A) $\angle$ B then BD is $\left\{\begin{matrix}<\\ >\end{matrix}\right\}$ Quadrant.	9
BP. BD.B	D	Firſt, find B A. by the 4th Caſe, Σ B. R : : τ B P. τ B A. Then by the 3d Rule S, D A. S, B A : : T, B. T, D. if B A. is $\left\{\begin{matrix}<\\ >\end{matrix}\right\}$ B D. then D is $\left\{\begin{matrix}\text{like}\\ \text{unlike}\end{matrix}\right\}$ B.	10

Th

The Actual Solution of the forementioned Caſes in Numbers.

Plate VI. Caſe 1. Given BP=34*d.*50′
Fig. IV. Keil. C. 9. DP=25 30
B=30 07 } Reqd. D.

S, DP. S, B : : S, BP. S, D.

S, B. 9.7004981
S, BP. 9.7567815

Sum 19.4572796
Subſt. S, DP. 9.6339844

Rem. 9.8232952 the Sine of D { 41*d.* 44′ / 138 16 if { Acute, / Obtuſe.

Plate VI. Caſe 2. Given BP=40*d.*00′
Fig. V. K. C. 10. B=80 00
D=47 10 } Reqd. PD.

S, D. S, BP : : S, B. S, DP.

S, BP. 9.8080675
S, B. 9.9933515

Sum 19.8014190
Subſt. S, D. 9.8653021

Rem. 9.9361169 the Sine of PD.=59*d.* 41′

Caſe 3. Given $\left.\begin{array}{r} \text{B P}=39d.30 \\ \text{P D}=36.\ 20 \\ \text{B}=24.\ 00 \end{array}\right\}$ Reqd. B D **Plate VI. *Fig.* VI.**

/k. c. 3.

Firſt, R. Σ B : : T, B P. T, B A.

Σ B. 9.9607302
T, B P. 9.9161045

Sum 19.8768347
Subſt. R. 10.0000000

Rem. 9.8768347 the Tangent of B A. 36*d.* 59′

Then by the 2d Rule.

Σ B P. Σ B A : : Σ D P. Σ D A,

Σ B A. 9.9024438
Σ D P. 9.9061107

Sum 19.8085545
Subſt. Σ B P. 9.8874061

Rem. 9.9211484 the Σ D A = 33*d.* 30′

Then the $\left\{\begin{array}{l} \text{Sum} \\ \text{Diff.} \end{array}\right\}$ of $\left\{\begin{array}{l} 36d\,59' \\ 33\ \ 30 \end{array}\right\} = \left\{\begin{array}{l} 70d\,29' \\ \ \ 3\ \ 29 \end{array}\right\}$ = B D, according as the Perpendicular falls, within or without the Triangle.

Caſe 4. Given the ſame as before. Required P.

K.C.7.

Firſt, ⋦, B P. R : : τ B. T, B P A.

R. 10.0000000
τ B. 10.3514169

Sum 20.3514169
Subſt. ⋦ P B. 9.8874061

Rem. 10.4640108 the Tan. of B P A 71*d.* 03′

Then by the 4th Rule,

T, D P. T, B P : : ⋦ B P A. ⋦ D P A.

T, B P. 9.9161045
⋦, B P A. 9.5115397

Sum. 19.4276442
Subſt. T, DP. 9.8665644

Rem. 9.5610798 the ⋦ D P A. 68*d.* 40′

Then the $\left\{\begin{matrix}\text{Sum}\\ \text{Diff.}\end{matrix}\right\}$ of $\left\{\begin{matrix}71d\,03'\\ 68\;\;40\end{matrix}\right\} = \left\{\begin{matrix}139d\,43'\\ 2\;\;23\end{matrix}\right\}$ = P. according as the Perpendicular falls within or without the Triangle.

Caſe 7. Given B P = 60*d.* 10 } Reqd. D *Plate* VI. *Fig.* IX.
K. C. 2. B = 40. 00
P = 118. 6

Firſt, Σ BP. R : : τ, B. T, B P A.

R. 10.0000000
τ B. 10.0761865

Sum. 20.0761865
Subſt. Σ B P. 9.6967745

Rem. 10.3794120 T, B P A = 67*d.* 21′

Then B P D — B P A = D P A = 50*d.* 45′.

Then by the Firſt Rule.

S, B P A. S, D P A : : Σ B. Σ D.

S, D P A. 9.8889612
Σ B. 9.8842540

Sum 19.7732152
Subſt. S, B P A. 9.9651426

Rem. 9.8080726 = Σ D = 50*d.* 00′.

Plate VI. *Fig.* X. Caſe 8. Given the ſame. Required D P.
K. C. 8.

Firſt, Σ B P. R : : τ B. T, B P A.

Which will be found as before, 67*d* 21′, and conſequently D P A. 50*d.* 45′.

Then by the 4th Rule,

Σ D P A. Σ B P A : : T, B P. T, D P.

Σ B P A. 9.5855745
T, B P. 10.2414830

Sum 19.8270575
Subſt. Σ D P A. 9.8012015

Rem. 10.0258560 T, D P. = 46*d.* 42′. and muſt be leſs than a Quadrant, becauſe D P A is of the ſame kind with B, that is of the ſame kind with P A, otherwiſe P D would have been more than a Qnadrant by the 5th *Solution.*

Plate VI. Caſe 9. Given B P = 60*d.* 10′ ⎫
Fig. XI. K. C. 4. B = 40 00 ⎬ Reqd. P D.
B D = 87 30 ⎭

Firſt, Σ B. R : : τ B P. τ B A.
Which will be found as in the 6th Caſe from the *Data* 53*d.* 11′, and Conſequently D A = 34*d.* 19′

Then by the 2d Rule,

Σ B A. Σ P B : : Σ D A. Σ D P.

Σ B P. 9.6967745
Σ D A. 9.9169455

Sum 19.6137200
Subſt. Σ B A. 9.7776128

Rem. 9.8361072 Σ D P = 46*d.* 43′.
Becauſe D A is like B, or P A otherwiſe it would have been greater than a Quadrant.

Caſe 5. Given BP = 60d. 10′ } Req. P. — *Plate* VI. *Fig.* VII.
B = 40. 00
D = 50. 00

R.C.1.

Firſt, Σ BP. R : : τ B. T, BPA.

R. 10.0000000
τ B. 10.0761865

Sum 20.0761865
Subſt. Σ BP. 9.6967745

Rem. 10.3794120 the Tan. of BPA. 67*d.* 21′

Then by the Firſt Rule,

Σ B. S, BPA : : Σ D. S, DPA.

S, BPA. 9.9651426
Σ D. 9.8080675

Sum. 19.7732101
Subſt. Σ B. 9.8842540

Rem. 9.8889561 S, DPA = 50 *d.* 45′

Then the Sum of { 67.21. / 50.45 } *viz.* 118.06 = P. ſince the Angles B and D are of the ſame kind; but if they had been of different kinds, the difference of theſe two Angles would have been equal to P.

Caſe 6. Given the ſame as before. Required B D.

K. G. F.

First, Σ B. R :: τ B P. τ B A.

R. 10.0000000
τ B. P. 9.7585170

Plate VI.
Fig. VIII.

Sum 19.7585170
Subſt. Σ B. 9.8842540

Rem. 9.8742630 the τ B A = 53d 11'

Then by the 3d Rule,

T, D. T, B :: S, B A. S, D A.

T, B. 9.9238135
S, B A. 9.9033923

Sum. 19.8272058
Subſt. T, D. 10.0761865

Rem. 9.7510193 S, D A. 34d. 19'

Therefore the Sum of $\left\{\begin{matrix}\text{B A } 53d. & 11'\\ \text{D A } 34 & 19'\end{matrix}\right\}$ *viz.* 87*d.* 30' = B D ſince the Angles B and D are of the ſame kind, otherwiſe their diff. = B D.

Therefore A E, A C :: (D L, D *m*) :: *b* L, β *m*, theref. A E, A C :: *b* L, β *m*, and that *m* F, α C :: β *m*, *y* α will be evident from the Conſtruction of the Figure, the two Right-angled Triangles B F β, ϖ C *y* are Similar, and therefore,

P F, F β :: ϖ C, *c y*
i. e. *m* F, F β :: α C, *c y*
theref. *m* F, *m* F — F β :: α C, α C — *c y*
i. e. *m* F, β *m* :: α C, *y* α.
and conſequently *m* F, α C :: β *m*, *y* α.

Caſe 11. The 3 Sides of any Spheric Triangle being given to find an Angle.

K.
C II.

S, *m* x S, *n*, R *q* :: S $\overline{\frac{1}{2}\text{ B} + \frac{1}{2}\text{ diff.}}$ *c r* x S, $\overline{\frac{1}{2}\text{ Baſe} - \frac{1}{2}}$ $\overline{\text{diff.}}$ *c r*. S *q* ½ Angle, Dem. AE x *m* F, R *q* :: (by the 4th Axiom) *b* L, *y* α :: *b* L x ½ R, *y* α x ½ R (that is by *Lemma* 5th and 1ſt) :: S $\overline{\frac{1}{2}\text{ Baſe} + \frac{1}{2}\text{ diff.}}$ *c r* x S $\overline{\frac{1}{2}\text{ Baſe} - \frac{1}{2}}$ $\overline{\text{diff.}}$ *c r*, S *q* ½ Angle.

Inſtance in Numbers.

Given, B P. 60d 10′
D P. 46 42
B D. 87 30
} Reqd. P.

½ the Baſe is	43d	45′
½ the diff. of the Sides is	6	44
½ Baſe + ½ diff. *c r*. is	50	29
½ Baſe — ½ diff. *c r*. is	37	01

R q.	20.0000000
$S\,\overline{\tfrac{1}{2}\,\text{Base}+\tfrac{1}{2}\,\text{dif. }cr} \times S,\,\overline{\tfrac{1}{2}\,\text{Base}-\tfrac{1}{2}\,\text{dif. }cr.}$	19.6669325
Sum	39.6669325
Subſt. S, $m \times$ S, n,	19.8002534
Rem.	19.8666791
S q, $\tfrac{1}{2}$ the Angle P, the $\tfrac{1}{2}$ of which	9.9333395

is the Sine of 59d 03′ $\tfrac{1}{2}$ the Angle required, which being doubled is 118d 06′ = P.

COROLL. I.

Suppoſe m, n, the Legs of the Angle required b the Baſe, $Z = m + n$, $X = m - n$. $\mathfrak{z} = \tfrac{1}{2}$ Sum of m, n, b.

Then, $S, m \times S, n, \; R\,q : : S, \overline{\mathfrak{z} - m} \times S, \overline{\mathfrak{z} - n} : S\,q\,\tfrac{1}{2}$ Angle.

Theref. $S\,q\,\tfrac{1}{2}\,\text{Angle} = \dfrac{R\,q \times S, \overline{\mathfrak{z} - m} \times S, \overline{\mathfrak{z} - n}.}{S\,m \times S\,n}$

Ulacq. preſcribes this following Rule, by two Proportions, which comes to the ſame;

$$S, m.\; S, \overline{\mathfrak{z} - m} : : S, \overline{\mathfrak{z} - n}.\; A. = \frac{S, \overline{\mathfrak{z} - m} \times S, \overline{\mathfrak{z} - n}}{S, m.}$$

$$\text{Then, } S\,n.\; R : : A.\left(\frac{S, \overline{\mathfrak{z} - m} \times S, \overline{\mathfrak{z} - n}}{S, m}\right).\; Q = \frac{R \times S, \overline{\mathfrak{z} - m} \times S, \overline{\mathfrak{z} - n}}{S, m \times S, n}$$

$$\text{Then } Q \times R = S\,q\,\tfrac{1}{2}\,\text{Angle} = \frac{R\,q \times S, \overline{\mathfrak{z} - m} \times S, \overline{\mathfrak{z} - n}.}{S, m \times S, n}$$

COR-

Caſe 10. Given as before to find D.

R. C. 6.

1. B A, as in the 9th and 6th Caſes, from the ſame *Data*, will be found 53*d.* 11′, and Conſequently D A = 34*d.* 19′.

Plate VI. *Fig.* XII.

Then by the 3d Rule.

S, D A. S, B A : : T, B. T, D.

S, B A. 9.9033923
T, B. 9.9238135

Sum 19.8272058
Subſt. S, D A. 9.7510991

Rem. 10.0761067 T, D. 50*d.* 03′
for here B A is < than B D, and conſequently B and D are like.

LEMMA V.

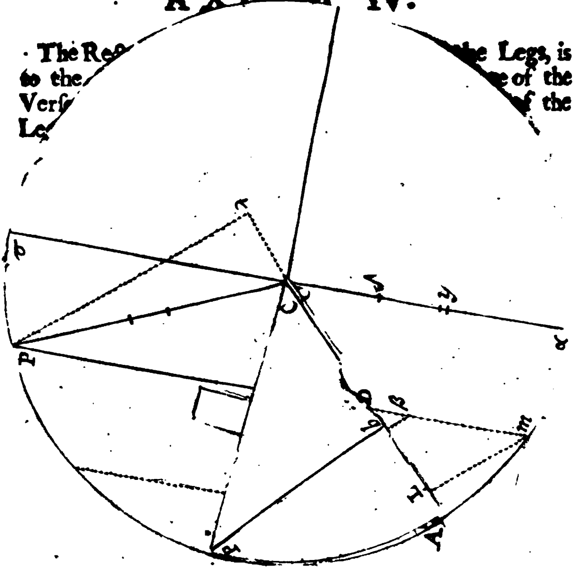
The difference of the Verſed-Sines of two Arcs × $\frac{1}{2}$ R, = S $\frac{1}{2}$ Sum of thoſe Arcs Multiplied by the Sine of the $\frac{1}{2}$ diff. of the Arcs.

Suppoſe the 2 Arcs A F, A E. the diff. of the Verſed-Sines, A G — A H = H G = E B the Sine of the $\frac{1}{2}$ Z of the Arcs is A D the Sine of the $\frac{1}{2}$ X of the Arcs in F O, now the Triangles A C D, E F B are Simillar, therefore A C, A D : : F E, F B or $\frac{1}{2}$ AC, A D : : $\frac{1}{2}$ F B (F O). F B, therf. $\frac{1}{2}$ A C × F B = F O × A D.

Plate VI. *Fig.* XIII.

AXIOM IV.

The Re … Legs, is
to the … of the
Verſ … the
Le …



Suppoſe B the Angle Required, B A and B P (= BM) its Legs, and ſuppoſe PA the Baſe to be Oblique to the Plane B A, then will P B A be an Oblique Triangle.

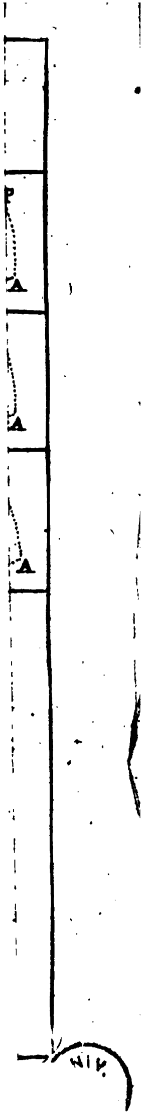
Let fall $P\beta$ perpendicular to βm and ϖy perpendicular to $C\alpha$, and Pb perpendicular to bA, and mL perpendicular to C A, therefore βb is perpendicular to C A, and $bL = bA - LA =$ V Baſe — V diff. Legs.

$$\left.\begin{array}{l}\text{Now } AE, AC :: bL, ϐm \\ \text{And } mF, \alpha C :: ϐm, y\alpha\end{array}\right\}$$ and multiplying the correſpondent Terms of both *Proportions*.

$AE \times mF, AC \times \alpha C :: (bL \times ϐm . ϐm \times y\alpha) :: bL, y\alpha.$

Now that $AE, AC :: bL, ϐm$ will be evident from the Similitude of the Triangles AEC, DLm, Dbϐ.

There-


P
A
A
A
NIV.

COROLL. II.

Putting μ = Compliment of m. *Gunter* reſolves the 11th Caſe by this ſingle Proportion.

$$S, m . \sigma n : : \Sigma B \backsim S, \overline{\mu + n} . V.$$

For by the fourth Axiom.

$$S, m \times S, n . r r : : V B \backsim V \overline{m - n} . V.$$

And dividing the 1ſt and 2d Term by $S\, n$,

$$\begin{array}{rl} \dfrac{S, m . r r}{S n}\left(= \sigma n\right) & : : V B \backsim V \overline{m - n} . V. \\ & : : \Sigma B \backsim \Sigma \overline{m - n} . V. \\ & : : \Sigma B \backsim S, \overline{\mu + n} . V. \\ \hline \text{Therf. } S, m . \sigma n & : : \Sigma B \backsim S, \overline{\mu + n} . V. \end{array}$$

The three laſt third Terms being only different Deſignations of the ſame Quantity. For that $V B \backsim V \overline{m - n} = \Sigma B \backsim \Sigma, \overline{m - n}$, is evident on Inſpection of the Figure; and that $\Sigma B \backsim \Sigma, m - n = \Sigma B \backsim S, \overline{\mu - n}$ will be evident, if we prove that $\Sigma \overline{m - n} = S, \overline{\mu + n}$, which may be done thus;

Suppoſe q = a Quadrant. Then the Complement of $\overline{m - n}$ is $q - m + n$, and in the room of μ, putting $q - m$, to which it is equal by *Hypotheſis*. $S, \mu + n = S, \overline{q - m + n}$, the Arcs be equal, the Sines are equal.

COROLL. III.

S *m* x S, *n* : R *q* : : V Z *c r* — V. Baſe, *v* of the Angle, being a Propoſition like the 4th Axiom.

For reſuming the laſt Figure, reduce the Plane B P to the ſame plane with B A, ſo that they may make one Circle, and let fall P λ perpendicular to A C, then *b* λ = λA—BA, that is V. Z *c r*—V. Baſe; but

A E. A C : : *b*λ. β P } and Multiplying the
and *m* F. R. : : β P. *v*. } Correſpondent Terms.

---

we have A E x *m* F. R *q* : : β P x *b* λ. β P x *v*. and dividing the 3d and 4th Term by β P.
we have A E x *m* F. R *q* : : *b* λ. *v*. Angle required.
or S *m* x S, *n*. R *q* : : *b* λ. *v*.

Now that A E. A C : : *b* λ. β P will be evident from the Similitude of the Triangles A E C, D *b* β, D P λ.

Therefore,

A E. A C : : D *b*. D β : : D λ. D P : : D*b* + D λ. D β + D P. *i. e.* *b* λ. D P.

Therefore A E. A C : : *b* λ. D P.

And that *m* F. R : : β P. *v*. will be evident from the Similitude of the Right-angled Triangles β F P, *y* C ϖ.

For F P. C ϖ : : F β. C *y* : : F P + F β. C ϖ + C *y*. *i. e.* : : β P. *y* ϖ.

and becauſe F P = *m* F. and C ϖ = R. and *y* ϖ = *v*.
Therefore *m* F. R : : β P. *v*. of the Angle.

This being prov'd; hence follows,

COR.

COROLL. IV.

$S, m \times S n. R q :: \overline{S. \tfrac{1}{2} Zcr + \tfrac{1}{2} \text{Baſe}} \times \overline{S, \tfrac{1}{2} Zcr - \tfrac{1}{2} \text{Baſe}}. \Sigma q \tfrac{1}{2}$ Angle.

For by the laſt Corol. $S m. \times S n. R :: b \lambda. v :: b \lambda \times \tfrac{1}{2} R.$

$v \times \tfrac{1}{2} R$ (*i. e.* by Lem. 5th and 2d) $:: \overline{S \tfrac{1}{2} Zcr + \tfrac{1}{2} \text{Baſe}} \times \overline{S, \tfrac{1}{2} Zcr - \tfrac{1}{2} \text{Baſe}}. \Sigma q. \tfrac{1}{2}$ Angle. Hence,

COROL. V.

$S m \times S n. R q :: S. \text{ʒ} \times S. \overline{\text{ʒ} - B}. \Sigma q \tfrac{1}{2}$ Angle.

COROL. VI.

$S, \text{ʒ} \times S, \overline{\text{ʒ} - B}. S, \overline{\text{ʒ} - m} \times S, \overline{\text{ʒ} - n} :: R q. T q \tfrac{1}{2}$ Angle.
For by the 5th and 1ſt Coroll.

$S, \text{ʒ} \times S, \overline{\text{ʒ} - B}. \Sigma q \tfrac{1}{2}$ Arch $:: (S m \times S n. R q) :: S, \overline{\text{ʒ} - m} \times S, \overline{\text{ʒ} - n}. S q \tfrac{1}{2}$ Angle.

Therefore $S, \text{ʒ} \times \overline{\text{ʒ} - B}. S, \overline{\text{ʒ} - m} \times S, \overline{\text{ʒ} - n} :: \Sigma q \tfrac{1}{2}$ Arch. $S q \tfrac{1}{2}$ Arch.
But becauſe $\Sigma. S :: R. T.$
Therefore,

$S, \text{ʒ} \times S, \overline{\text{ʒ} - B}. S, \overline{\text{ʒ} - m} \times S, \overline{\text{ʒ} - n} :: R q. T q \tfrac{1}{2}$ Arch.

COROLL. VII.

$S, \overline{\text{ʒ} - m} \times S, \overline{\text{ʒ} - n}. S, \text{ʒ} \times S, \overline{\text{ʒ} - B} :: R q. \tau q \tfrac{1}{2}$ Angle,
This follows from the laſt, and becauſe $T, R :: R. \tau.$

COROLL. VIII.

$$V Z - V X . 2 R :: \overline{V B - V X} . V L.$$ Or

$$:: V Z - V B . v . L.$$

Which is the Practice of *Foſter*, with his Line of Verſed Sines.

DEMONSTRATION.

By the *Fourth* Axiom.

$V B - V X . V L :: S, m \times S, n . R q ::$ (by *Cor.* 3.) $V Z - V B . v L ::$ the Sum of the firſt and fifth Terms. Sum of the ſecond and ſixth. *i. e.* $:: V Z - V X$ $2 R.$

COROL. IX.

$$\frac{\Sigma Z \backsim \Sigma X}{2} . R. :: \Sigma B \backsim \Sigma X . V L.$$ or

$$\frac{\Sigma Z \backsim S. \overline{\mu + n}}{2} . R :: \Sigma B \backsim S, \overline{\mu + n}\, V . L.$$

This follows from the *eighth Corollary*, and may be eaſily underſtood from what is ſaid in the Explication of the *ſecond Corollary*.

But here note, That if the Sum of *m* and *n* be $\left\{\begin{array}{l}\text{leſs}\\ \text{greater}\end{array}\right\}$ than a Quadrant, the Σ of the Sum muſt be underſtood as $\left\{\begin{array}{l}\text{Affirmative}\\ \text{Negative}\end{array}\right\}$ and added, or ſubſtracted accordingly.

CASE

Caſe XII. K. C. 12.

The Three Angles being given, to find a Side.

The Angles adjacent to the Side required, call Legs, and the Angle oppoſite call the Baſe; then work as in the 11th Caſe.

For ſuch is the Operation in the Supplemental Triangle, whoſe Angles and Sides are equal to the Supplements of the Sides and Angles of the Triangle propoſed; and Arcs and their Supplements have the ſame Sines and Tangents.

Of the ſeveral Methods for the Solution of the 11th *Caſe, mentioned in the Nine Corallaries from the* 4th *Axiom; the* Reader *need only commit ſome one of them to Memory for his conſtant Uſe: Such as* Cor. I. *and* Cor. V. *The reſt being mentioned only that the* Reader *may know the Reaſon of the ſeveral Methods made uſe of by different Authors.*

So likewiſe the Lemma's *and Rules following are not to be thought of Abſolute Neceſſity, ſince the Solution of all the Caſes may be ſolved by what has been done before, they being (as before) inſerted to give the Reaſon of other Rules for the ſame Purpoſe; which a* Theoriſt *would not willingly be ignorant of.*

LEM-

LEMMA VI.

The Sum of the Sines of two Arcs, is to the Difference of the *Sines*, as the Tangent of half the Sum of the Arcs is to the Tangent of half the Difference of them.

Z, Sines 2 Arcs. X. Sines : : T, $\frac{1}{2}$ Z Arcs. T, $\frac{1}{2}$ X Arcs.

Let the 2 Arcs be E A, F A (in the Fig. to *Lem.* V.) then 'twill be,

$$EG + FH.\ EG - FH :: \frac{EG + FH}{2}.\ \frac{EG - FH}{2}$$

i. e. vu. oj : : *o*L. *o*F : : *n*M. *n*K.

Theref. *bu. oj* : : *n*M. *n*K. q. e. d.

LEMMA VII.

Z, Co-ſines. X, Co-ſines : : *τ* $\frac{1}{2}$ Z Arcs. T, $\frac{1}{2}$ X Arcs. If the Arcs are leſs than 90 Degrees. For in the forementioned Figure, the 2 Arcs being A E, A F. G C and C H are their Co-ſines ; it will be

$$CH + CG,\ CH - CG :: \frac{CH + CG}{2}.\ \frac{CH - CG}{2}.$$

*i. e. cu. u*H : : *w*O. *o*F : : *π*N. NK.

Therf. *cu. u*H : : *π*N. N K.

But if the Arcs are bigger than 90 Degrees,

Z, Co-ſines. X, Co-ſines : : T, $\frac{1}{2}$ Z Arcs. *τ* $\frac{1}{2}$ X Arcs.

The Tangents of 2 Arcs being reciprocally proportional to their Co-tangents, by the *third Lemma*.

LEMMA

LEMMA VIII.

Z, Tangents. X, Tangents : : S, Z Arcs. S. X Arcs.

Let the Arcs be N A. N E.

P *m*. K *m* : : E L. F L : : E G. F H.
Theref. P *m*. K *m* : : E G. F H. Q. E. D.

LEMMA IX.

$$S,\ Arc = \frac{S,\ \frac{1}{2}\ Arc \times \Sigma\ \frac{1}{2}\ Arc}{\frac{1}{2}\ R}$$

For in *Fig*. 10. *Plate* the 1ſt, A E. F *n* : : B *n*. ½ F B = ½ R. the Triangles A E B, F *n* B, are ſimilar.

$$\text{Theref.}\quad A\,E = \frac{B\,n + F\,n}{\frac{1}{2}\,F\,B} = \frac{S,\ \frac{1}{2}\ Arc \times \Sigma\ \frac{1}{2}\ Arc}{\frac{1}{2}\ R}$$

LEMMA X.

The Sines of two Arcs A, E, are proportional to the Rectangles, under the Sines and Co-ſines of their ½ Arcs,

$$S,\ A.\ S,\ E\ : :\ (\text{by the } \textit{ninth Lemma})\ \frac{S,\frac{1}{2}A \times \Sigma\ \frac{1}{2}A.}{\frac{1}{2}R.}\ \frac{S,\frac{1}{2}E \times \Sigma\ \frac{1}{2}E}{\frac{1}{2}R.}\ : :\ S,\frac{1}{2}A \times \Sigma\ \frac{1}{2}A.\ S,\frac{1}{2}E \times \Sigma\ \frac{1}{2}E.$$

LEMMA XI.

The Tangents of two Arcs A, E, are proportional to the Rectangles under the *Sines* and *Secants* of thoſe Arcs.

i. e. T, A. T, E : : S, A × ſ, A. S, E × ſ, E.

For T, A. T, E : : $\frac{S, A \times R}{\Sigma A}$. $\frac{S, E \times R}{\Sigma E}$: : S, A, × Σ E × R.

S, E × Σ A × R : : S, A × Σ E. S, E × Σ A : : S, A × ſ, A.
S, E × ſ, E.

For by the *fourth Lemma*, the Co-ſines of two Arcs are reciprocally proportional to their Secants.

RULE V.

The Tangents of the Baſes are proportional to the Tangents of the Angle at the Vertex.

Plate V.
Fig. xviii. For by the *ſecond* Axiom,

T, *L* B P A. T, B A (: : R. S, P A) : : T, D P A. T, D A.
Therſ. T, B P A. T, B A : : T, D P A. T, D A.

RULE VI.

S, Z, *C r*. S, X. *C r* : : τ $\frac{1}{2}$ Z *L L* at the Vertex.
T, $\frac{1}{2}$ X *LL*, at the Vertex : : τ $\frac{1}{2}$ X *LL*, at the Vertex.
T, $\frac{1}{2}$ Z *LL*, at the Vertex.

DEMONSTRATION.

For, by the *fourth* Rule,
T, BP. T, DP : : Σ, D P A. Σ, B P A.

Therefore,

$T, BP + T, DP . T, BP - T, DP :: \Sigma, DPA + \Sigma, BPA . \Sigma DPA - \Sigma BPA.$

That is, by *Lemma* VIII, and VII.

As $S, Z\,Cr . S, X\,Cr :: T, \frac{1}{2} Z\,LL$, at the Vertex. $T, \frac{1}{2} X\,LL$, at the Vertex.

RULE VII.

$T, \frac{1}{2} Z\,cr . T, \frac{1}{2} X\,cr :: T, \frac{1}{2} Z\,LL$, at Baſe. $T, \frac{1}{2} X\,LL$, at Baſe.

Therefore,

$T, \frac{1}{2} Z\,cr . T, \frac{1}{2} X\,cr :: T, \frac{1}{2} X\,LL$, at Baſe. $T, \frac{1}{2} Z\,LL$, at Baſe.

DEMONSTRATION.

For, by the *third* Axiom,

$S, BP . S, DP :: S, D . S, B.$

Therefore,

$S, BP + S, DP . S, BP - S, DP :: S, D + S, B . S, D - S, B.$

Therefore, by the *ſixth Lemma*,

$$T, \frac{BP + DP}{2} . T, \frac{BP \backsim DP}{2} :: T, \frac{D + B}{2} . T, \frac{D \backsim B}{2}$$

RULE VIII.

$T, \frac{1}{2} Z\,LL$, Vertex. $T, \frac{1}{2} Z\,LL$, Baſe $:: T, \frac{1}{2} X\,LL$, Baſe. $T, \frac{1}{2} X\,LL$, Vertex.

For, by the *firſt* Rule.

$S, BPA . S, DPA :: \Sigma B . \Sigma D.$

Therefore,

$$\text{S, BPA}+\text{DPA}.\ \text{S, BPA}-\text{S, DPA} :: \Sigma\,\text{B}+\Sigma\,\text{D}.\ \Sigma\,\text{B}-\Sigma\,\text{D}.$$

Therefore by *Lemm.* VI, VII.

$$\text{T, }\frac{\text{BPA}+\text{DPA}}{2}.\ \text{T, }\frac{\text{BPA}\backsim\text{DPA}}{2} :: \tau,\frac{\text{B}+\text{D}}{2}.\ \text{T, }\frac{\text{B}\backsim\text{D}}{2}.$$

$$\text{Theref. }\ \text{T, }\frac{\text{B}\backsim\text{D}}{2}.\ \text{T, }\frac{\text{BPA}\backsim\text{DPA}}{2} :: \left(\tau\,\frac{\text{B}+\text{D}}{2}.\ \text{T, }\frac{\text{BPA}+\text{DPA}}{2}\right) :: \tau\,\frac{\text{BPA}+\text{DPA}}{2}.\ \text{T, }\frac{\text{B}+\text{D}}{2}.$$

P R O B L E M I.

In any Sphæric Triangle B P D. *Given the Angle* P, *and the Sides* B P, D P. *To find the two Angles at the Baſe.*

S, $\frac{1}{2}$ Z *cr*. S, $\frac{1}{2}$ X *cr* :: τ $\frac{1}{2}$ P. T, $\frac{1}{2}$ X *LL*, Baſe.
And Σ $\frac{1}{2}$ Z *cr*. Σ $\frac{1}{2}$ X *cr* :: τ $\frac{1}{2}$ P. T, $\frac{1}{2}$ Z *LL*, Baſe.

DEMONSTRATION I.

Rule 6. S, Z *cr*. S, X *cr* :: τ $\frac{1}{2}$ P. T $\frac{1}{2}$ X *LL*, Vertex.
Rule 7. T, $\frac{1}{2}$ Z *cr*. T, $\frac{1}{2}$ X *cr* :: T $\frac{1}{2}$ Z *LL*, Baſe, T $\frac{1}{2}$ X *LL*, Baſe,
Rule 8. T, $\frac{1}{2}$ Z *LL*, Baſe, T $\frac{1}{2}$ X *LL*, Vertex :: τ $\frac{1}{2}$ P T, $\frac{1}{2}$ X *LL*, Baſe.

Therefore, Multiplying the Homologous Terms into each other, and Dividing the 1ſt and 3d Products; as likewiſe the 2d and 4th by what is an equal Component in each ; and we have,

S, Z *cr.* x T $\frac{1}{2}$ Z *cr.* S; X *cr* x T $\frac{1}{2}$ X *cr* : ; τ *q* $\frac{1}{2}$ P,
T *q* $\frac{1}{2}$ X *LL*, Baſe.

Therefore (by *Lemma* I.)

S *q* $\frac{1}{2}$ R *cr.* S *q* $\frac{1}{2}$ X *cr* : : τ *q* $\frac{1}{2}$ P. T *q* $\frac{1}{2}$ X *LL*, Baſe.

And therefore the Roots of theſe Quantities are Proportional.

S, $\frac{1}{2}$ Z *cr.* S $\frac{1}{2}$ X *cr* : : τ $\frac{1}{2}$ P. T $\frac{1}{2}$ X *LL*, Baſe, Q. E. D.

DEMONSTRATION II.

By Rule 6. S, Z *cr.* S, X *cr* : : τ $\frac{1}{2}$ P. T $\frac{1}{2}$ X *LL*, Vertex,
Rule 7. τ $\frac{1}{2}$ Z *cr.* τ $\frac{1}{2}$ X *cr* : : T $\frac{1}{2}$ X *LL*, Baſe,
T, $\frac{1}{2}$ Z *LL*, Baſe.
Rule 8. T $\frac{1}{2}$ X *LL*, Baſe, T $\frac{1}{2}$ X *LL*, Vertex : ; τ $\frac{1}{2}$ P.
T $\frac{1}{2}$ Z *LL*, Baſe.

Therefore, Multiplying the Homologous Terms into each other, and Dividing the 1ſt and 3d Products ; as likewiſe the 2d and 4th by what is an equal Component in each, we have,

S, Z *cr.* x τ $\frac{1}{2}$ Z *cr.* S, X *cr* x τ $\frac{1}{2}$ X *cr* : : τ *q* $\frac{1}{2}$ P.
T *q* $\frac{1}{2}$ Z *LL*, Baſe.

Therefore (by *Lemma* II.)

Σ *q* $\frac{1}{2}$ Z *cr.* Σ *q* $\frac{1}{2}$ X *cr* : : τ *q* $\frac{1}{2}$ P. T *q* $\frac{1}{2}$ Z *LL*, Baſe.

Therefore,

Σ $\frac{1}{2}$ Z *cr.* Σ $\frac{1}{2}$ X *cr* : ; τ $\frac{1}{2}$ P. T $\frac{1}{2}$ Z *LL*, Baſe. Q. E. D.

This Demonſtration ſuppoſeth P = Z *LL*, at Vertex. that is, that the Perpendicular falls within the Triangle ; but the ſame Proportions will ariſe, if it falls without.

PROB.

PROBLEM II.

In any Sphæric Triangle let there be given two Angles, with the interjacent Baſe (or ſide included between thoſe Angles ; To find the other two Sides, and the remaining Angle.

$S,\frac{1}{2}Z\,LL.\ S\,\frac{1}{2}X\,LL :: T\,\frac{1}{2}$ Baſe. $T,\frac{1}{2}X$ *cr.*
$\Sigma\,\frac{1}{2}Z\,LL.\ \Sigma\,\frac{1}{2}X\,LL :: T\,\frac{1}{2}$ Baſe. $T,\frac{1}{2}Z$ *cr.*

The Demonſtration may be perform'd by the ſame *Lemma*'s and Rules that the preceding was ; but will be needleſs, if you change the laſt Triangle into its Supplemental.

RULE IX.

In any Sphæric Triangle having let fall a Perpendicular Arc from the Vertex to the Baſe, it will be

$T\,\frac{1}{2}Z$ Segm. of the Baſe. $T\,\frac{1}{2}Z$ *cr* $::$ $T\,\frac{1}{2}X$ *cr.* $T\,\frac{1}{2}$ X Segm. of the Baſe.

For, by the *ſecond* Rule,
$\Sigma BP.\ \Sigma DP :: \Sigma BA.\ \Sigma DA.$
Therefore,
$\Sigma BP + \Sigma DP.\ \Sigma BP \backsim \Sigma DP :: \Sigma BA + \Sigma DA.\ \Sigma BA \backsim \Sigma DA.$
Therefore, by Lemma VII.
$\tau\,\frac{1}{2}Z$ *cr.* $T\,\frac{1}{2}X$ *cr* $::$ $\tau\,\frac{1}{2}Z$ Segm. $T\,\frac{1}{2}X$ Segm.
Therefore,
$T\,\frac{1}{2}X$ *cr.* $T\,\frac{1}{2}X$ Segm. ($::$ $\tau\,\frac{1}{2}Z$ *cr.* $\tau\,\frac{1}{2}Z$ Segm.) $::$ $T\,\frac{1}{2}Z$ Segm. $T\,\frac{1}{2}Z$ *cr.*

PROB.

PROBLEM III.

Suppoſe there are given the Three Sides of a Spherical Triangle, to find an Angle. Having imagin'd a Perpendicular let fall to make two Right-angled Triangles.

T $\frac{1}{2}$ Baſe. T $\frac{1}{2}$ Z *c r* : : T $\frac{1}{2}$ X *c r*. T, F.

Then F $\underset{\backsim}{+}$ $\frac{1}{2}$ Baſe $=$ $\left\{\begin{matrix}\text{greater}\\ \text{leſſer}\end{matrix}\right\}$ ſegment, and if F $\underset{<}{>}$ $\frac{1}{2}$ Baſe, the Perpendicular falls $\left\{\begin{matrix}\text{without}\\ \text{within}\end{matrix}\right\}$ the Triangle.

Then, R, τ B P : : T, B A. ∠ B,
R, τ D P : : T, D A, ∠ D.

FINIS.

OF

Stereographic Projection

OF THE

SPHERE.

Stereographic Projection is a Branch of *Perſpective*, which teacheth us how to draw the Appearances of the Circles of the Globe, upon a Plane paſſing thro' the Centre thereof; ſuppoſing the Eye in Surface of the Globe in the Pole of the Circle on which the Projection is to be made. And here the firſt thing that ought to be done, is to prove that the Repreſentation of all Circles deſcribed upon the Surface of the Globe (whether greater or leſſer) ſhall be Circles upon the Plane of the Projection. In order to which I premiſe this

LEMMA.

Plate VII, *Fig.* I.

If a Scalene Cone, A B C, whoſe Baſe is a Circle, be cut by a Plane paſſing thro' the Vertex A, and Center of the Circle of the Baſe; making the Triangle A B C, (the Plane of which Triangle being perpendicular to the Plane of the Baſe of the Cone. I ſay, That if this Cone be ſuppoſed to be cut by two other Planes (perpendicular to the Plane of the Triangle) the one parallel to the Baſe as F G, the other E D, ſo as to make the Triangle A E D, ſimilar to the Triangle A C B. the Curvities F H G, E H D, ſhall be Circles, as is proved by *Apollonius*, *Mygdorgius*, and other Writers of *Conics*.

PROP. I.

If from the Eye at A, plac'd in the Globe's Superficies, there be infinite Right Lines imagin'd to be produced to the Periphery of any Circle (ſmall or great) B C, deſcribed on it, thereby making a conic Superficies. I ſay, the common Interſection of the Surface of this Cone, with the Plane of the Circle E G (in whoſe Pole the Eye is placed) or any other Plane D C, parallel to it, ſhall be a Circle.

Plate VII. *Fig.* II, III.

DEMONSTRATION.

The Triangle A B C, is the Section of the Cone thro' its Vertex and Centre of the Baſe. Now if we prove that the Triangles A B C, and A C β, are ſimilar, the Propoſition will be manifeſt.

Now the Arc D A = C A. therefore the Angle A B C = A C β, by the 20th of the *third* of *Euclid*; and the Angle A, common to both Triangles, therefore the Triangles A B C, and A C β, are ſimilar. And therefore by the *Lemma*, the common Sections, C β, and *c b*, which is parallel to it, ſhall be Circles.

COROL. I.

From hence 'tis evident, That if the Circle to be projected, be parallel to the Plane on which the Projection is drawn, its Repreſentation will be a Circle; for the Cone is cut by a Plane parallel to its Baſe.

II. If it be any way Oblique to the *Plane* of the *Projection*,

jection, its Repreſentation will be a Circle, ſince the Cone will be cut according to the *Lemma*.

III. But if the Eye be in the *Plane* of any Circle, its Repreſentation will be a right Line, the Lines drawn from the Eye to the *Periphery* being in the ſame *Plane*, and the common Section of two *Planes* is a right Line.

IV. The Line *b c* is the projected Diameter of the Circle B C. the Line E G in which is the projected *Diameter*, I call with Mr. *Oughtred*, the *Line of Meaſures*.

P R O P. II.

Plate VII. *Fig.* IV.

If a great Circle be to be projected upon the Plane of another great Circle, the Center of the projected Circle ſhall lie in the Line of Meaſures, diſtant from the Center of the primitive Circle (or Circle on which the Projection is drawn) by the Tangent of the Elevation of the ſaid Circle above the Plane of the Projection.

DEMONSTRATION.

Suppoſe the Eye at A, projecting the great Circle B C, upon the plane of the Circle E G; the Point B, appears in the Line of *Meaſures* at *b*. ſo that F *b* is the Tangent of ½ the Angle H F B, the diſtance of the Point B, from the Vertex, by the 20*th* of the *third* of *Euclid*, and C, the other Extremity of the *Diameter* of the Circle B C, appears in the *Line of Meaſures* at *c*, ſo that the Diſtance F *c*, is the Tangent of of ½ the Angle H F C, the Diſtance of the Point C, from the Vertex, the Line *b c* being biſſected in *i*, *i* is the Center.

Now the right-angled Triangle *b* A *c*, is divided by the perpendicular A F, into two right-angled Triangles, *b* A F, A *c* F, ſimilar to each other by the 8th of the 6th Book of *Euclid*.

Therefore the $\angle\, b\,A\,F = \frac{1}{2}\,B\,F\,H = A\,c\,F = \frac{1}{2}\,A\,i\,b$, Therefore the Angle $B\,F\,H = A\,i\,b$, and conſequently their Complements are equal, *viz.* $BFE = FAi$. Therefore Fi, the Diſtance of the Center of the projected Circle from the Center of the primitive Circle is equal to the Tangent of the Arch EB, the Elevation of the Circle to be projected to the *Radius* AB of the Primitive Circle, *q. e. d.*

Mr. *Oughtred* determining only b the Appearance of the Extremity of the Diameter above the Plane of the Projection, gives this Rule for deſcribing the Circle.

From H the Pole of the Line of Meaſures make $Hm = BE = mn$, and draw the Lines Am, An, p ſhall be the projected Pole of BC and i its Center. The *Radius* being $ib = iA$. which he proves thus, the Angle $ibA = iAb$.

Becauſe the Angle $ibA = bBF + BFb$ that is $FAb + FAi$, for the Arc $Hn = 2\,BE$ by Conſtruction. Hence Fi, is evidently the Tangent of BE. and AI, its Secant $= bi$.

COROL.

Hence the Secant of the Circles Elevation ſet off in the Line of Meaſures from b, the projected Extremity of the Diameter above the Plane of the Projection will give i its Center, as well as the Tangent of that Elevation from the Center.

PROP. III.

If a leſſer Circle whoſe *Poles* lie in the *Plane* of the *Projection* were to be projected; the Center of its Repreſentation on the *Plane* ſhall be in the Line of Meaſures diſtant from the Center of the *Primitive Circle* by the Secant of that leſſer Circles diſtance from its *Pole*, and its Semidiameter ſhall be equal to the Tangent of that Diſtance.

Plate VIII. *Fig.* I.

DEMONSTRATION.

Suppoſe the leſſer Circle paſſing through the *Points* B, C, equally diſtant from the *Pole* G, were to be projected upon the *Plane* E G, to the Eye at A. Draw the Lines A B, A C, interſecting the Line of Meaſures in *b* and *c*, the Repreſentation of the two Extremities of the Diameter, biſſect the Line *b c* in *i*, then is *i* the Center, upon the Point *i* with the Diſtance *i b*, deſcribe the Circle *i* B *b* C *c*, draw the Lines H B *c*, F B, and *i* B.

The two Triangles A H B, and *c* H F, having the Angle at H Common, and the Angles at B and F right, are ſimilar.

Therefore $\angle HAB = \angle HcF$
but $\angle HAB = \frac{1}{2} HFB$
and $\angle HcF = \frac{1}{2} biB$.

Therefore $\angle HFB = \angle biB$. Therefore the Tri-Angle F B *i* is right angled at B. and conſequently B *i* is the Tangent of the Arc, B G the Diſtance of the leſſer Circle from its *Pole*, and F *i* its Secant. Q. E. D.

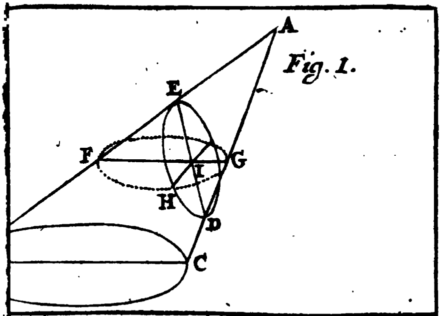
Mr. *Oughtred* only makes uſe of *b* the Repreſentation of B, and then directs to draw the Radius F B, and B *i* perpendicular to it, touching the Circle B G. proves *i* to be the Center, the Radius being $ib = iB$, thus
the $\angle ibB = \angle iBb$
For the $\angle b = FBb + BFb$ that is $= bBC + CBi = bBi$.

PROP. IV.

If two Circles C B F, and A B I, interſect each other, the Angle F B I, form'd by them at their Interſection, ſhall be equal to the Angle C B A, made by the Radius C B, A B, drawn to the point of their Interſection B.

DEMON-

Fig. 1.



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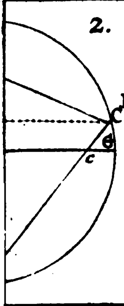

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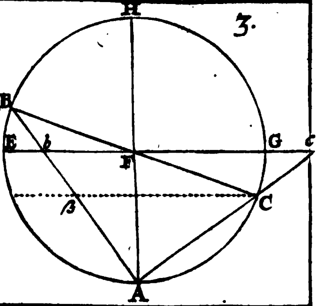

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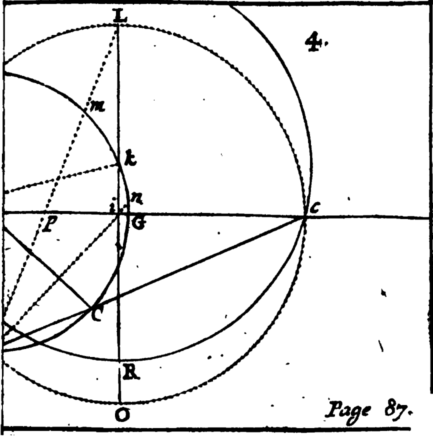

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DEMONSTRATION.

To the Point of Interſection B, draw the Line D B. a Tangent to the Circle C B F, and E B a Tangent to the Circle A B I. Now becauſe the infinitely ſmall Portions of the Circles C B F, aud A B I do coincide with the Tangents D B, and B E, and conſequently have the ſame Direction, therefore the Curve lined Angle F B I = to the right lined Angle formed by the Tangents D B E. Now the Angle C B D = L = A B E, and taking away E B C which is common to both, there will remain C B A = D B E *i e* = F B I. *q. e. d.*

Plate VIII. *Fig.* II.

PROP. V.

In *Stereographic Projection* the Angles made by the Circles on the Surface of the Sphere are equal to the Angles made by their Repreſentatives on the Plane of the Projection.

DEMONSTRATION.

Suppoſe the Eye at A, projecting the Angle R B S, upon the Plane E F: Let B D, be a Tangent to the Circle B S and B C, to the Circle B R, to the common Point B: The Plane E F, and likewiſe the Plane made by the Tangents B D, and B C, are both perpendicular to the *Plane* of the Circle B Q A, and their common Section D C, will therefore be perpendicular to the Line E F, produced.

Plate VIII. *Fig.* III.

The Eye at A, projects the Tangent B D in F D, and the Tangent B C in F C.

I am now to prove that the Angle C B D, made by the Tangents to theſe Circles on the Sphere, is equal to the Angle C F D, made by the projected Tangents on the *Plane* of the *Projection*. Draw B Q parallel to E F, and joyn A Q.

(a) By 32. 3. *Euclid.*
(b) 21. 3.
(c) 29. 1.
(d) *Ax.* 1.
(e) 6. 1.
(f) 4. 1.

For the Angle D B A =(a) A Q B, =(b) A B Q =(c) B F D =(d) D B F. Theref. B D =(e) D F.

Then in the Triangles C D F, C D B, C D, D F = C D, D B, and the Angles C D F, and C D B, both Right. Therefore C F D =(f) C B D. *q. e. d.*

Another Demonſtration by Mr. Halley.

Plate IX. *Fig.* I.

Let E B P L, be any great Circle of the Sphere, E the Eye placed in its Circumference, C its Center, P any point thereof; and let F C O be ſuppoſed a *Plane* erected at Right Angles to the Circle E B P L, on which F C O the Sphere is to be projected. Draw E P croſſing the *Plane* F C O in *p*, and *p* ſhall be the Repreſentation of the point P; to the point P, draw the Tangent A P G, and on any point thereof, as A, erect a Perpendicular A D, at Right Angles, to the *Plane* E B P L, and draw the Lines P D, A C, D C: Then the Angle A P D ſhall be equal to the Spherical Angle contain'd between the *Planes* A P C, D P C; draw alſo, A E, D E, interſecting the *Plane* F C O, in the points *a* and *d*, which will be the Repreſentation of the points A, and D; join *a d*, *p d*. I ſay, by the Triangle *a d p*, is ſimilar to the Triangle A D P, and the Angle *a p d*, equal to the Angle A P D.

Draw P L, A K, parallel to F O, and by reaſon of the Parallels *a p*. *a d* : : A K. A D. but (by the 32*d.* of the *third* of *Euclid*) in the Triangle A K P, the Angle A K P = L P E = A P K = E P G: Therefore the Side A K = A P. and *a p*. *a d* : : A P. A D. Hence (the Angles D A P, *d a p*, being Right) the Angle A P D = *a p d*; that is, The Spherical Angle on the Surface, is equal to that on the Projection. *q. e. d.*

PROP.

PROP. VI.

All leſſer Circles whoſe *Poles* lie not in the plane on which the *Projection* is to be made, are projected, by laying off in the Line of Meaſures from the Center, the Tangent of half the Diſtance of each Extremity of the Diameter from that Pole (of the Circle on which the *Projection* is made) oppoſite to the Eye, and the Diſtance of theſe two projected Extremities biſſected will be the Center of the Circle in the Projection.

For Example.

In *Plate* VII. *Fig*. II. Suppoſe B C to be ſuch leſſer Circle ; Lay off F *b* equal to the Tangent of half the Arch H B, and F *c* equal to the Tangent of half the Arch H C ; this Diſtance *b c* being biſſected, ſhall give the Center of that Circle in the *Projection*. The Reaſon of this is evident from what hath been ſaid in the *Demonſtration* of the *firſt Propoſition*.

PROP. VII.

All great Circles (as *Plate* VII. *Fig*. IV.) paſſing through the Point *b*, ſhall have their Centers in the Right Line *i* L produced ; which Right Line *i* L, is at Right Angles to the Line of Meaſures E G, paſſing through *i*, the Center of the Circle A *b* H, which cuts the Line E G, or its Repreſentative Semicircle, at Right Angles.

For all great Circles paſſing through *b*, ſince they will cut again at a Semicircles diſtance, muſt paſs thro' the projected oppoſite point, that its through *c* ; therefore 'tis evident their Centers will be in the Line *i* L produced, if need be.

PROP. VIII.

If it were required to draw a Circle which ſhall paſs through the point *b*, and make any given Angle with the Circle A *b* H, ſuppoſe of 20 *deg.* Lay off *i k*, equal to the Tangent of 20 *deg.* to the Radius *i b*; I ſay, the point *k* is the Center required.

For draw the Line *k b*, and deſcribe the Circle *b* R *c.*

Then by the *fourth Prop.*

The Angle A *b* R = *i b k.* = 20 *deg.*

COROL.

Since it has been proved in the *fifth Propoſition*, that the Angles made by the Circles on the *Plane* of the *Projection*, are equal to the Angles made by their Repreſentatives on the Surface of the Sphere : It will therefore follow, that if *b* was the projected Pole of the World, A *b* H, the Projection of the Six a Clock Hour-Circle, and it were required to project all the other Hour-Circles, which make Angles of 15, 30, 45 *deg.* &c. with that Six a Clock Hour-Circle; 'tis but putting off in the Line *i* L, from *i*, on both ſides the Tangent of 15, 30, 45, 60, 75. to the Radius *i b*, and you will have the Centers of all thoſe Hour-Circles in ſuch Horizontal Projection.

From what hath been demonſtrated in theſe *Eight Propoſitions*, 'twill be eaſy to project the Sphere in any Poſition : I will now proceed to ſhew how to meaſure any Arch projected, whether Great or Small, in order to which I premiſe theſe *Lemma's.*

LEMMA I.

If a *Plane* paſs through the *Poles* of two great Circles Æ Q, and H O, its common Section with the Surface of the Sphere will be repreſented by the great Circle Z P N H; and if the two remoteſt Poles, P and N, be connected by the right Line P N, and a *Plane* be ſuppoſed to re-

Plate IX.
Fig. II.

NIV. OF MICH.

volve on this Line P N as an Axis; and let the common Section of this revolving *Plane* with the Surface of the Sphere be repreſented by the Circle P D K N. I ſay, that this revolving Plane ſhall always cut off equal Arches upon thoſe Circles Æ Q, H O, whoſe Poles are thus connected between it ſelf P D K N, and the great Circle Z P N H; that is, the Arch D O = Arch K Q. And likewiſe from any two parallel Circles, *b o*, and *æ q*, equally diſtant from the ſaid Poles N and P. that is, *d o* = *k q*.

For P O = N Q, and the Angle O P D = Q N K, it being the Angle of the Inclination of the two Planes, and the portion of the leſſer Circle P D = N K: Therefore theſe two Triangles being in all caſes qualified alike, will be equal; and therefore D O = K Q; and for the ſame Reaſon the Triangles *o* P *d* and *q* N *k* are equal, and conſequently *d o* = *k q*. Q. E. D.

LEMMA II.

Plate IX. *Fig.* III.

If a *Plane* paſs through the *Poles* of two great Circles, Æ Q and H O, its common Section with the Surface of the Sphere will be repreſented by the great Circle Z P N H; and if the two neareſt *Poles*, N and *p*, be connected by the right Line N *p*, and a Plane be imagin'd to revolve on this Line *N p*, as an Axis, 'tis plain that its common Section with the Surface of the Sphere will be a Circle, and is repreſented by the Circle *p*, *i*, *r*, *s*, *t*, *f*, *m*, *n*, *t*, N. I ſay this plane ſhall always cut off equal Arches in thoſe great Circles Æ Q, H O, whoſe Poles are thus connected; and from any other Parallels to them, *æ q*, *b o*, equally remote from *p*, and N, between it ſelf *p s*, *t*, *m*, *t*, N, and the great Circle Z P N A; but on the contrary ſide, that is, Æ S = M O. *æ t* = *f o*. *b r* = *n q*. H *i* = *t* Q. For the Triangles *p* Æ S and N O M, *p æ t* and N *o f*, *p b r* and N *q n*, *p* H *i* and N Q *t*, may, as before, be ſhewn to be exactly qualified alike. And therefore, &c. Q. E D.

I now proceed to give ſome Inſtances in the Actual Projection of the Globe upon the plane of the Horizon, and of the ſeveral Ways of Meaſuring the Arches in the Projection ſo projected; depending on the two foregoing *Lemma's.*

Plate X.

Suppoſe I would project the Surface of the Sphere upon the Plane of the Horizon, for the Latitude of 51 deg. 30 min.

Deſcribe the Circle N E S W, repreſenting the Horizon; through Z, the Center, draw the Line S Z N, and E Z W, at right Angles; the firſt repreſenting the Meridian of the place, or North and South Azimuth; the other, the prime Vertical or Azimuth of Eaſt and Weſt. Now becauſe the Northern *Pole* of the World is diſtant from the Zenith (or oppoſite *Pole* to which the Eye is placed) 38 *deg.* 30 *min.* Lay off the Tangent of its half, *viz.* 19 *deg.* 15 *min.* in the Line of Meaſures S N, from Z to *p*; then will *p* be the Repreſentation of the Northern Pole of the World, by the *Eighth Propoſition*; and the Circles which paſſes through the points E *p* W, will be the Repreſentation of the Hour-Circle of Six; and its Elevation above the *Plane* of the projection being equal to the Elevation of the Pole, *viz.* 51 *deg.* 30 *min.* Lay off in the Line of Meaſures, Z T, the Tangent of that Circle's Elevation: Then will T be the Center of that Circle, by the *ſecond Propoſition*; then deſcribe the Circle E *p* W.

Now to deſcribe the Equator, whoſe Diſtance from the *Zenith* is 51 *deg.* 30 *min.* Take the Tangent of its half, *viz.* 25 *deg.* 45 *min.* to the Radius Z W (which you may eaſily have by the help of a Sector) and ſet it off in the Line of Meaſures from Z to *æ*, and the Equator ſhall paſs through the Meridian N S, in the point *æ*.

Now the Elevation of the Equator being always equal to the Complement of the Latitude, which in this Caſe is 38 *deg.* 30 *min.* Take the Tangent of its Elevation, and put it off in the Line of Meaſures from Z to X. Then ſhall X be the Center of the Equator. Then deſcribe the Circle E *æ* W, which you may compleat in an occult Arch, for a Uſe hereafter to be mention'd.

Now for drawing the Tropic of *Cancer*. Take the Tangent of half its neareſt and greateſt Diſtance from the Zenith, and put it off in the Line of Meaſures, from Z to *n*, and from Z to *r*; the Diſtance *n r* being biſſected, the point of Biſſection ſhall be the Center, by the *ſixth Propoſition*. Then deſcribe the Circle ♋ *n* ♋, which compleat in an occult Arch for farther Uſe.

After the ſame manner deſcribe ♑ *s* ♑, the Tropic of *Capricorn*: And the like Method uſe for the Deſcription of any other parallels to the Equator.

Now to draw the Hour-circles which make Angles of 15, 30, 45, &c. with the Hour-circle of Six, it hath been ſhewed in the *Cor.* 8*th Propoſition*, that their Centers will all lie in the Line P T P, which is perpendicular to the Line of Meaſures paſſing through the Center of the Hour-circle of Six: And that the Center of any Circle paſſing thro' the Point *p*, and making an Angle therewith, was to be found by laying off in the Line T P, the Tangent of that Angle, to the Radius *p* T. Therefore put off in the Line T P on both ſides the Tangent of 15, 30, 45, &c. and you'll have the Centers of all the Hour-circles. But to avoid confuſion of Lines in the Scheme, I have only put off the Tangent of 30 *deg.* from T to *t*, and thereby deſcribed one of them, *viz.* the Hour-circle of Four in the Afternoon.

Now if 'twere required to divide W *æ*, which repreſents the Quadrant of the Equator, into its Degrees: Lay your Ruler upon *p*, the projected Northern Pole of the Equator, and the equal Diviſions of the Quadrant of the Horizon W S. for inſtance, upon 30 and 60 *deg.* and the Ruler ſhall cut the Quadrant of the Equator W *æ*, in the points 30 and 60. So that *æ* 30, repreſents 30 *deg.* *æ* 60, repreſents 60 *deg.* And the like for any intermediate Degrees.

Or a Ruler laid upon (or a Line drawn from) *π*, the projected Southern Pole of the World, and the equal Diviſions of the Quadrant of the Horizon W N, ſhall paſs through the ſame Points 30, 60, &c. in *æ* W, the Repreſentative of the Weſtern Quadrant of the Equator.

Now the Reaſon of this will be evident, for in this Projection the Eye is ſuppoſed in the under-Pole of the Horizon or Nadir, and a Line drawn from the Eye to the Northern Pole of the World, paſſes thro' the Line of Meaſures, S N in *p*, the Repreſentation of that Northern Pole. Let this imaginary Line which paſſes from the Eye to the real Northern Pole, be the Line that connects the two remoteſt Poles of the Equator and Horizon, according to the *firſt Lemma*; and imagine a *Plane* to revolve upon that Axis; this *Plane* ſhall according to this *Lemma*, cut off equal Arches in thoſe great Circles whoſe Poles are thus connected between it ſelf and the Meridian N S, which paſſes through the two Poles of the Horizon, and two Poles of the Equator. Now the Eye in the Nadir is in the *Plane* of this revolving *Plane*; and therefore by the *third Corollary* of the *firſt Propoſition*, the Repreſentation of that Circle which is the common Section of this revolving *Plane* with the Surface of the Sphere, will be repreſented to the Eye in the common Section of this revolving *Plane*, with the *Plane* of the Projection; that is, (if this *Plane* paſſes through 30 in the Quadrant S W) in the Line *p*, 30. if it paſſes thro' 60 *deg.* in the Quadrant S W, the Section is *p*, 60. therefore *æ* 30, in the Equator, is equivalent to S 30, in the Horizon, *&c.* For this revolving *Plane* always cuts the Line of Meaſures in *p*, the Projection of the Northern Pole.

But when I divide the Quadrant *æ* W, by Lines drawn from π, the projection of the Southern Pole to the equal Diviſions of the Quadrant W N. The Line upon which the Revolution of the *Plane* is made, is a right-Line drawn from the Eye ſuppoſed in the Nadir, to the Southern Pole, and produced till it meet the Line of Meaſures N S produced in π, the Repreſentation of the Southern Pole; but by the *ſecond Lemma*, a *Plane* that revolves upon this Line, cuts off equal Arches in the Quadrant of the Equator *æ* W, and Quadrant of the Horizon W N. And therefore a Line drawn from π to 30, in the Quadrant N W, ſhall paſs through 30 in the Repreſentation of the Quadrant of the Equator *æ* W.

Now

Plate 9th

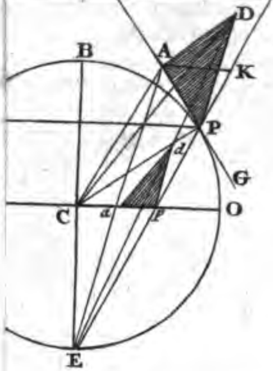

fig. 1

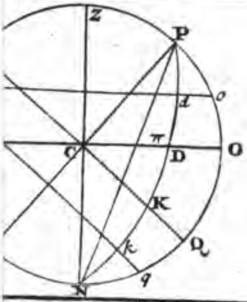

fig. 2

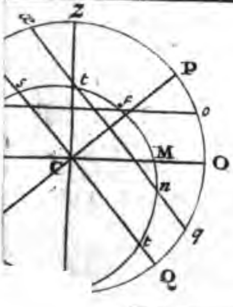

fig. 3

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Now to divide the Tropic of *Cancer*.

Firſt, If I would do it by means of the projected North Pole *p*, ſince the Tropic of *Cancer* is diſtant from the North-Pole 66 *deg.* 30 *min.* from the North-Pole there muſt be a *Parallel* to the Horizon projected, that is 66 *deg.* 30 *min.* diſtant from the under Pole of the Horizon, in which the Eye is placed, its Diſtance from the upper Pole of the Horizon is 113 *deg.* 30 *min.* Take the Tangent of its half, *viz.* 56 *d.* 45 *m.* and put it on the Line of Meaſures from Z to A, Z A ſhall be the Semidiameter of this *Parallel*; or the point A may be determin'd by putting the Chord of 66 *d.* 30 *min.* from E to *m*, a Line drawn from E through *m*, will meet the Line of Meaſures in the ſame point A. Deſcribe then the Circle A B C D, a Ruler laid from *p*, to the equal Diviſions of this *Parallel*, ſhall cut the correſpondent Points or Degrees in the Tropic of *Cancer*. For inſtance: A Line drawn from *p* to 30. in the Quadrant A D, ſhall cut the portion of the Tropic of *Cancer n* ♋, in 30: So that *n* 30, ſhall repreſent 30 *deg.* *n* 60, ſhall repreſent 60 in the Tropic, &*c.*

Secondly, If I would divide this Tropic by the means of the projected Southern-Pole *π*; I conſider that the Tropic of *Cancer* is diſtant from the Southern-Pole of the Æquator 113 *d.* 30 *m.*

I therefore project a parallel to the Horizon as far diſtant from the under *Pole* of the Horizon, and conſequently from the upper *Pole* of the Horizon, or Zenith, 66 *d.* 30 *m.* Therefore I put off the Tangent of its half, *viz.* 33. *d.* 15 *m.* in the Line of Meaſures, from Z to *a*; or put the Chord of 66, 30, from W to *w*, a Line drawn from E to *w*, ſhall cut the Line of Meaſures in the ſame point *a*: Deſcribe the Circle *a b c d*, ſuppoſe the Quadrant *c d*, divided, a Ruler laid, or a Line drawn from *π* to 30, 60, 90, in the Quadrant *c d*, ſhall cut the portion of the Tropic of *Cancer* in the points 30, 60, 90.

Now thoſe *Parallels* A B C D, *a b c d*, which were drawn for the Diviſion of the Tropic of *Cancer*, will divide the Tropic of *Capricorn*, *viz*, Lines drawn from to the equal Diviſions of the Semicircle *a b c d*, will divide the Tropic ♑ *s* ♑.

Or Lines drawn from *π*, to the equal Diviſions of the Arch F C H, will do the ſame. I have omitted drawing them, leſt I ſhould too much perplex the Scheme. The ſame is to be underſtood of any other two *Parallels* equally diſtant from the Æquator, or any other Circle. For having firſt drawn two *Parallels* to the Horizon, or other Circle on which the *Projection* is made, by the fore-mentioned Rules, that are neceſſary to meaſure one of your parallel Circles, by Lines drawn from both Poles: The ſame two *Parallels* will ſerve to meaſure or divide the other parallel Circle at an equal diſtance. Only here obſerve (as in the Diviſion of the Tropic of *Capricorn*) that that *Parallel* to the Horizon, which ſerved to divide the firſt *Parallel* from the projected North-Pole, muſt be uſed to divide the *Parallel* at an equal Diſtance from the projected South-Pole.

The Equator W *æ*, beſides the former Ways given for its Diviſion, may be thus divided: Suppoſe the Semicircle F G H, was divided into 180 equal Degrees, a Ruler laid upon any of theſe Diviſions, and the projected Northern-Pole; as ſuppoſe from 60, or *v*, in the Quadrant G H, to *p*, it ſhall cut the Equator *æ* W, in 60, which I thus demonſtrate.

Let the Lines 60 *p* 60, 60 X, Z 60, be drawn, as alſo X E, and *p* E. Now becauſe that Z X, is the Tangent of the Elevation of the Equator above the Horizon, *viz.* 38, 30. and Z *p*, the Tangent of half that Angle: Therefore in the Triangle E Z X, the Line E *p*, divides the Angle Z E X, into two equal parts. Therefore by the *third* of the *ſixth* of *Euclid*, the Line Z X is divided in the point *p*. So that Z *p*. *p* X. : : Z E. E X. but Z E = Z *y*, and E X = X *v*. Therefore Z *p*. *p* X : : Z *y*. X *v*. but the Angle *v p* X = Z *p y*. Therefore by the *ſeventh Propoſition* of the *ſixth* of *Euclid*, the two Triangles *p* X *v*, and *p* Z *y*, are ſimilar: And therefore the Angle *p* X *v* = *p* Z *y*. In each of which, taking away a Right Angle, the Remainder H X *v*, and *y* Z W, are equal; And conſequently their Compliments

G X *v*,

G X *v*, and Y Z S are equal: And therefore this Method gives the ſame Point 60, in the Equator *α* W; as if a Line had been drawn from the Pole *p*, to the point 60, or *y*, in the Quadrant of the Horizon S W. Or draw Lines from any of the Diviſions in the Semicircle F G H, to *π*, the projected Southern Pole, ſhall divide the Equator. For inſtance; Lines drawn from 30, 60, 90, in the Quadrant F G, to *π*, paſs through the Points 30, 60, 90, in the Quadrant of the Equator *α* W, as you may ſee in the Scheme, and may be demonſtrated by a Demonſtration more eaſy than the former.

The Tropic of *Cancer* may be divided likewiſe as the Equator was, *viz*. by dividing the Semicircle *α r β*, into 180 Deg. and drawing Lines from thoſe Points thro' the Point *p*, and they will divide the Tropic of *Cancer*. Or Lines drawn from thoſe Points to *π*, the Southern-Pole will do the ſame; as you may ſee in the Scheme, Lines drawn from the Points 30, 60, 90, in the Quadrant *r β* to *π*, paſs thro' the Points 30, 60, 90, in the Tropic of *Cancer*. The ſame Method may be uſed for the Diviſion of any Parallels; The Demonſtration being like the former, I therefore omit it.

OF THE Orthographic Projection OF THE SPHERE.

ORthographic Projection ſuppoſes the Eye perpendicular to a Plane paſſing through the Center of the Sphere, poſited in a Perpendicular to that Plane erected from the Center, but at an *infinite Diſtance*; or which comes to the ſame thing, ſuppoſes that there were Perpendiculars let fall from each Point, to the Circles to be projected to the Plane, on which the Projection is to be made: Theſe projected Points will fall in Right-Lines on the Plane, when the Circles (great or ſmall) to be projected, are perpendicular to the *Plane* of the *Projection*; but if *Oblique*, in *Ellipſes*. So that you may have as great Variety of theſe ſorts of *Projections*, as in the *Stereographic*. But they being more Troubleſome, by reaſon of the Oblique Circles being projected in *Ellipſes*; Mathematicians have confined themſelves to the moſt ſimple of theſe kind; *viz*. upon the Plane of the Solſticial Colure, commonly called the *Planiſphere*, or *Analemma*, which is admirably uſeful and eaſy. I ſhall here give you a ſhort Account of it.

In

In *Plate* 11th, *Fig.* 1ſt, the Circle Z H N O, repreſents the Solſticial Colure, or Meridian paſſing through the 1ſt Points of ♋ and ♑, the Center A repreſents the Eaſt or Weſt Point of the Horizon.

H A O the Horizon, its Axis Z A N, repreſents the Azimuth of Eaſt or Weſt, or 90th Azimuth.

Æ A *æ* is the Equinoctial, its Axis P A *p*, repreſents the 6th Hour Cricle.

♋ A ♑ repreſents the Ecliptic, ♈ A ♎, its Axis, repreſenting a Circle paſſing thro' the Equinoctial Points ♈ and ♎.

O P the heighth of the Pole, whoſe Compliment H Æ = Z P, is the height of the Equinoctial.

Suppoſe the Sun's place to be in ♊ 0 *deg.* or ♌, 0 *deg.* or in ♐ 0 *deg.* or in ♒ 0 *deg.* theſe Points are all diſtant from the Equinoctial Points (both which are projected in A) 60 *deg.* therefore take the Sine of 60 *deg.* and put it off in the Line A ♋, and A ♑, from A, both Ways to ☉, ☉; theſe Points ſhew the place of the Sun for the given times. *Viz.* That in the Quadrant, A ♋ for the Summer, and that in the Quadrart, A ♑ for the Winter; and through the Point ☉, draw a Parallel to the Equinoctial D *d*, δ *δ*, which is the Projection of the Sun's Parallel for that Day.

Hence it appears, that the Sun that Day riſes or ſets in the Point M or *m*, Culminates at Noon in D or δ, comes to the 6th Hour in T, is due Eaſt or Weſt in Q, therefore A M, or A *m* is the Sine of the Sun's Amplitude, T ☉ is the like Right Aſcenſion of the Sun, that is, T ☉ contains as many Degrees as the Right Aſcenſion, had it been meaſured in the Equator, and therefore a like Arch M D or M δ the Semidiurnal Arch of the Sun, M T or *m t* the Aſcenſional Difference, to be numbred from the next Cardinal Point backward or forward.

Suppoſe the Sun's height were taken by a Quadrant, or other Inſtrument, put that Altitude from H to L, and draw L X *l* Parallel to the Horizon, cutting the Parallel of Declination D *d* in S, and the Axis of the Horizon in X, S will be the place of the Sun in the Analemma for that inſtant. Wherefore S T ſhall be t like Diſtance of the Sun from the 6th Hour Circle, S X its like Diſt. from the 90th Azimuth, whence may be found in theſe Parallels, both the Hour at which the Obſervation was made, and Azimuth from the Meridian.

Thoſe Arches of leſſer Circles (which I call like Arches) may proportionally be transferred to their greateſt Parallel in this manner : As the Semidiameter of the leſſer Parallel, is to the like diſtance : So is the Semidiameter of the Circle Z H N O, to the Sine of the true Diſtance from the Center. For Inſtance, in the like Aſcenſional Difference T M to the end of the Parallel T M *d*, draw a Semidiameter A *d*, and within the Triangle A T *d*, draw from M, the Right Line M *π*, parallel to A T ; Then T *d*. T M : : A *d*. A *π*, make A F = A *π*, then is A F the Sine of the Aſcenſional Difference, by which the Semidiurnal Arch is more or leſs than ſix Hours.

Hence it is eaſy to deſcribe the *Ellipſis* that ſhall paſs through the given Point M, in any *Ratio* of the Semitranſverſe A O, to the Semiconjugate A M. Thus,

> Make A O. A M : : X *l*. X G : : *q* P. *qy*, the Curve Line, drawn with an even Hand thro the Points Z *y* G M, ſhall be the *Ellipſis* required.

Or thus by your Sector ; Take A O in your Compaſſes, and open your Line of Sines to it, then take A M and apply it to your Line of Sines, and obſerve what Angle 'tis the Sine of ; then draw as many Occult Parallels to A O as you pleaſe, aud making each of them Rad. ſucceſſively, lay off the Sine of the Angle you obſerved A M to be the Sine of, and do ſo ſucceſſively, a Line drawn through theſe Points ſhall be what was deſired.

Stereo-

Stereographic Problems,

In order to the Projection *of the ſeveral Caſes in* Spheric Triangles.

PROB. I.

To find the Pole of any great Circle Projected.

NOTE that the Pole of any great Circle is the repreſentation of that point of the Surface of the Globe (on the Plane of the Projection) which is 90 *deg.* diſtant from that Circle.

Theſe Circles are of three kinds, either *Plate* XI. *Fig.* II.

1. The *Primitive Circle* A B C D, the Plane of which is the Plane of the *Projection.*

2. A *Right Circle* (or the projected Repreſentation of a great Circle perpendicular to the Plane of the Projection) the Eye being in the Plane of which, will on the projection be repreſented by a Diameter, as B D or A C.

3. *An Oblique Circle,* (or the projected Repreſentation of a great Circle not perpendicular to the Plane of the Projection) which will be always repreſented by an Arch, ſuch as B F D; the points of interſection of this Arch, with the primitive Circle (being connected) will always paſs through the Center of the primitive Circle.

For this Arch B F D is the Repreſentation of a Semicircle Elevated above the Plane of the Projection ; and ſince two great Circles cut one another at a Semicircle's diſtance, their common Section will be a Diameter, and here repreſented by the Diameter B D.

CASE I.

To find the Pole of the Primitive Circle, A B C D.

RULE.

Find its Center E, by the 1ſt *Prop.* of the 3d Book of *Euclid*, which point E, is the Pole.

CASE II.

To find the Pole of a Right Circle, ſuppoſe B D.

RULE.

Croſs that Diameter with another Diameter A C at Right Angles, the two Extremities A and C are the Poles of the right Circle B D. Or the Chord of 90 *deg.* ſet off from B both ways, will determine the Points A and C.

CASE III.

To find the Pole of an Oblique Circle.

Note, that the Pole of an Oblique Circle is in that Diameter of the primitive Circle, which paſſes through the Center of the projected Oblique Circle.

EXAMPLE.

$$\left.\begin{array}{l}\text{A B C D the Primitive}\\ \text{B F D the Oblique}\end{array}\right\}\text{Circle and}\left\{\begin{array}{c}\text{E}\\ \text{C}\end{array}\right\}\text{their Centers given.}$$

To find the Pole of the Oblique Circle B F D.

1. Through E and C, draw the Diameter A E C, cutting the primitive Circle in A and C, and the Oblique Circle in F.

2. A Ruler laid on D and F, will cut the primitive Circle in *a*.

3. Take 90 *deg.* from a Schale of Chords, and put it off in the primitive Circle from *a* to *b*.

4. A Ruler laid on D, and this point *b*, will cut the Diameter A E C in G, the Pole required; which point G is the projected Pole that lyes above the Plane of the Projection, but if the projected under Pole had been required, lay off the Chord of 90 *deg.* from *a*, the contrary way to *c*, a Ruler laid from D to *c*, will cut the Diameter A E C produced in the point γ, the projected under Pole.

PROB. II.

To lay down on the Projection any Angle required.

DEFINITION.

A Spherical Angle is made by the Intersection of 2 great Circles on the Surface of the Sphere, and is Measured by that Arch of a great Circle which is 90 *deg.* from this point of Intersection, intercepted between the Legs or Arches that constitute the Angle. It hath been proved in the 5th Propesition of Stereographic Projection, that the Angles of the plane of the Projection are equal to the Angles they represent on the Surface of the Globe.

In this Problem there are three Cases. Either,

1. Where the angular Point is at the Center of the primitive Circle.

2. Where the Angular Point is in the Periphery of the primitive Circle.

3. Where the Angular Point is in any point given within the Periphery, and not in the Center,

CASE I.

To make an Angle where the Angular Point is in the Center of the Primitive Circle.

RULE.

Such Angle is made in all reſpects like a plane Angle, *viz.* by drawing 2 Lines from the Center, ſo that the Arch of the primitive Circle intercepted between them, be equal to the Degrees of the Angle propoſed.

CASE II.

To make an Angle, that the Angular Point may be in a Point given in the Periphery.

Plate XI. *Fig.* II. For Example to make the Angle H A D, having its angular Point in the Periphery at A of 36 *d.* 00′.

RULE.

The Primitive Circle A B C D *being deſcribed.*

1. Lay a Rule on E, the primitive Circle's Center, and the given Point A, which will likewiſe cut the primitive Circle in C.
2. With the Secant of the given Angle 36 *deg.* and one Foot of the Compaſſes in the given point A, deſcribe the Arch X.
3. With the ſame opening of the Compaſſes, and one Foot in C, croſs the former Arch, X in X. X is the Center of the Oblique Circle A H C, making

king an Angle with the primitive Circle of 36 *deg.* as is evident by the *Coroll.* 2 *Propoſition* of *Stereographic Projection.*

CASE III.

To make an Angle where the angular Point is in any Point given within the Periphery, but not in the Center.

EXAMPLE.

A B C E the primitive Circle }
E its Center } given.
G the Point }

To deſcribe an Angle at the Point G, of 45 deg.

RULE.

1. Through G and E, draw the Diameter A E G C.
2. Croſs it at Right Angles with the Diameter B D.
3. Lay a Ruler on D and G, and it will cut the primitive Circle in *b*.
4. Take the Tangent of the Arch C *b*, and put it off in the Line A C, from E to A.
5. Erect a Perpendicular A *g* continued.
6. Lay off the Tangent of the Compliment of the Angle requir'd from A, either way to *g*, which Compliment is in this Caſe 45 *deg.*
7. Upon the Point *g* with the diſtance *g* G, deſcribe the Circle G *a*; the Angle A G *a*, is the Angle required.

This follows from the 8*th Propoſition* of *Stereog. Project.*

PROB. III.

To draw a great Circle through any given Point, making with the Primitive Circle any given Angle.

RULE.

RULE.

1. With the Tangent of the given Angle, and one Foot in the Center deſcribe an Arch.
2. With the Secant of the ſame Angle, and one Foot in the given Point, ſtrike an Arch croſſing the former; the Interſection of which Arches is the Center of the Circle required.

EXAMPLE.

Plate XI. A B C D the primitive Circle }
Fig. II. E the Center thereof } Given.
I the Point }

Through the Point I, 'tis required to draw an Oblique Circle, ſo that it may make an Angle at the primitive Circle of 36 *deg*.

1. With the Tangent of 36 *deg*. and one Foot in the Center, deſcribe the Arch X.
2. With the Secant of this Angle, and one Foot in the given Point, I make an Arch, cutting the Arch X in X, X is the Center of the Oblique Circle A H C, making with the primitive Circle the Angle required.

The reaſon of which is evident from the 2d *Propoſition* of *Stereog. Projection.*

PROB. IV.

To draw a great Circle paſſing through any 2 Points given within the Periphery of the Primitive Circle.

RULE.

1. Through any one of the given Points and the primitive Circle's Center, draw a Diameter produced beyond the Periphery.
2. Croſs this Diameter at Right Angles.

3. Through the Point mention'd, draw a Line to the Extremity of this ſecond Diameter.
4. At the end of this Line in the Periphery of the primitive Circle, erect a Perpendicular cutting the firſt Diameter in a third Point.
5. Through the two Points given, and this third Point ſtrike a Circle, and it ſhall be the great Circle requir'd, by *Prob.* 8. of *Geomet. Probl.*

EXAMPLE.

A B C D the primitive Circle }
E its Center } Given. *Plate* XII. *Fig.* I.
F and G the two Points }

Through F *and* G *'tis required to draw a great Circle.*

1. Through E and F draw the Diameter A F E C I.
2. Croſs it at Right Angles with the Diameter B D.
3. Draw a Line either from F to B, or from F to D.
4. To either of which Lines F B or F D, at the Extremities B or D, erect the Perpendiculars B I or D I, interſecting the Diameter A C produced in I, the 3d Point.
5. Through theſe 3 Points F, G, and I, deſcribe the Circle K F G H, and 'tis done.

We here have nothing to prove, but that the Line H K, that connects the 2 Interſections with the primitive Circle, paſſes through the Center; for then 'tis the Repreſentation of a great Circle.

DEMONSTRATION.

F E, E B : : E B, E I, the Triangles F E B and B E I, being ſimilar by the *8th* Prop. of the *6th* of *Euclid*, and H E continued will fall upon K, the other interſection; for, if it's poſſible to do otherwiſe, let it fall upon on M, and produce H E M to L, then by the 35th of the 3d of *Euclid.* $EF . EH :: EL . EI$, and conſequently $EH \times EL = EBq = EH \times EK = EH \times EM$, and by Conſequence $EM = EL$; that is, the part equal to the whole; which is abſurd. *Q. E. D.*

PROB. V.

To draw a great Circle perpendicular to a given great Circle.

General Rule.

Draw a great Circle through the Pole of the given great Circle, and it will be perpendicular to a a great Circle given. In this Problem are 4 Caſes, either

1. Perpendicular to the primitive Circle.
2. A Right Circle perpendicular to a Right Circle.
3. An Oblique Circle perpendicular to a Right Circle.
4. An Oblique Circle perpendicular to an Oblique Circle.

CASE I.

To draw a Circle perpendicular to the primitive Circle given.

RULE.

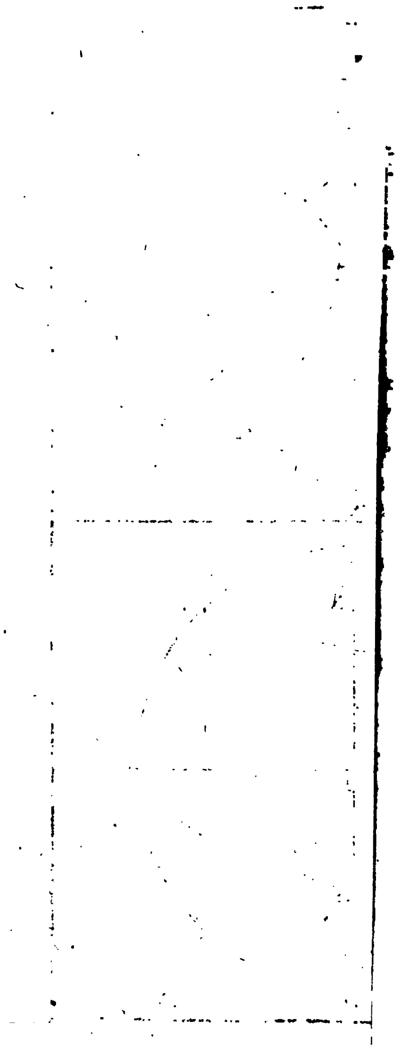
Through the Center of the primitive Circle draw a Diameter and 'tis done; for the Center of the primitive Circle is its Pole.

CASE II.

To draw a Right Circle perpendicular to a Right Circle given.

RULE.

This is done by drawing a Diameter, perpendicular to the Diameter or Right Circle given.



CASE III.

To draw an Oblique Circle perpendicular to a Right Circle given.

RULE.

1. Find the Poles of the given Right Circle, by Prob the 1ſt, Caſe the 2d, of *Stereog. Prob.*
2. Draw a Circle through thoſe two Poles.

EXAMPLE.

A B C D the primitive Circle } Given. *Plate* XI.
E its Center } *Fig.* II.
B E D a Right Circle }

'Tis required to draw an Oblique Circle perpendicular to the Right Circle B E D given.

1. Croſs the Right Circle given B E D, with the Diameter A C, or lay the Chord of 90 *deg*. from B to A and C, which are the 2 Poles of the right Circle B E D.
2. With any Diſtance, and one Foot in A and C, ſtrike 2 Arches croſſing one another in X. X ſhall be the Center, and A H C, the Circle requir'd.

But here obſerve, That if the Oblique Circle required to be drawn, were ſo limited as to make a given Angle with the primitive A D C.

Then take the Secant of that Angle in your Compaſſes, and placing your Compaſſes in A and C, make 2 Arches croſſing one another in X; X ſhall be the Center. Or take the Tangent of that Angle and ſet it off in the Right Circle (or Diameter) B D, from the Center E to X, and 'tis done.

But if the Point H be given, through which this Oblique Circle ſhould paſs, without any relation to the Angle it ſhould make with the primitive Circle, (though that be Virtually given) a Circle ſtruck thro' the 3 Points A, H, and C, will anſwer the Demand.

CASE IV.

To draw an Oblique Circle perpendicular to a given Oblique Circle.

RULE.

1. Find the Pole of the given Oblique Circle, by Prob. 1. Caſe the 3d. of *Stereog. Problems.*
2. Through that Pole draw a great Circle, or which is the ſame, draw ſuch an Arch as may paſs thro' the Pole ſo found, and may interſect the primitive Circle in Points diametrically oppoſite.

Plate XII. *Fig.* II.

EXAMPLE.

AHCG the primitive Circle	given.	I	their
ABC an Oblique Circle		Y	Centers.

'Tis required to draw an Oblique Circle perpendicular to the Oblique Circle ABC given.

1. To find E the Pole of the Oblique Circle given by Prob. 1ſt. Caſe 3d.
2. Lay a Ruler over the Center I, any-wiſe, cutting the primitive Circle in D and G, a Circle ſtruck through the three Points D, E and G, ſhall be the great Circle required.

But here obſerve, That if the Point K (in the given Oblique Circle) be given, then draw a great Circle thro' the 2 Points E and K, by the 4th Problem. And if it be required that the Circle paſſing thro' the Pole E, ſhall alſo make a given Angle with the primitive Circle, it may be done by the 3d. *Prob.*

PROB.

PROB. VI.

To lay any quantity of Degrees on a great Circle.

In this Problem are three Caſes.

1. On the primitive Circle.
2. On a Right Circle.
3. On an Oblique Circle.

CASE I.

To lay off any number of Degrees on the primitive Circle.

RULE.

Take from your Line of Chords (to that Radius) the number of Degrees you would lay off, and tranſfer it to the primitive Circle, and 'tis done. This needs no Example.

CASE II.

To lay off any number of Degrees on a Right Circle.

RULE.

Take the Tangent of half the Angle you would lay off, and put it off in the Right Circle from the Center, and 'tis done.

EXAMPLE. *Plate* XI. *Fig.* II.

A B C D the primitive Circle, E its Center } given.
A E C the Right Circle }

On this Right Circle A E C, 'tis required to lay off from the Center E 45 *deg.*

Take the Tangent of its half, *viz.* 22 *d.* 30′, and lay it off from E to F, then ſhall E F repreſent 45 *deg.*

Or from the Scale of half Tangents take 45 *deg.* and it will reach from E to F.

Or take the Chord of 45 *deg.* and lay it off in the Semicircle A B C, from B to *a*, a Ruler laid from D to *a* will cut the Right Circle in F, ſo that E F ſhall repreſent the Arc *a* B, *viz* .45 *deg.*

CASE III.

To lay off any number of Degrees upon an Oblique Circle.

RULE.

1. Find the Pole of the given Oblique Circle, by *Prob.* 1. *Caſe* 3.
2. Lay the given Number of Degrees on the primitive Circle, by *Caſe* the 1ſt of this Problem.
3. Lay a Ruler to the extremities of the Arch laid down in the primitive Circle from the Pole, and it will cut off a like Arch in the Oblique Circle.

Plate XI. *Fig.* II.

EXAMPLE.

A B C D the primitive Circle } their Centers { E }
B F D the Oblique Circle } are { C }

T'is required in the Oblique Circle B F D, to lay off an Angle of 45 *deg.* from F.

1. Through C and E the two Centers, draw the Diameter C E F A, and therein find G the Pole of the Oblique Circle given, by the 1ſt *Prob.* *Caſe* 3.
2. From your Line of Chords, put 45 *deg.* off in the Primitive Circle, from A to *a*.
3. Lay a Ruler to G and *a*, and it ſhall cut the Oblique Circle in *a*, ſo that F *a* repreſents 45 *deg.*

The like may be done for any other Number of Degrees, and from any Point.

Or thus, Lay off 45 *deg.* in the oppoſite Semicircle B C D, from C to *b*, a Ruler laid from γ the projected *under Pole*, of the Oblique Circle B F D to *b*, will cut the Oblique Circle given B F D in the ſame Point *a*. So that any Arch or Number of Degrees, may be laid off in any great Circle given, by the means of either Pole; the Demonſtration of this is already given in the Theoric part of *Stereog. Projection.*

PROB. VII.

To Meaſure any part of a great Circle projected. In this Problem are three Caſes, and are but the Converſe of the former.

CASE I.

To Meaſure any Part of the primitive Circle.

RULE.

Take the Arch required to be meaſured in your *Compaſſes*, and apply it to your Scale of *Chords*, and it ſhews you the Degrees; this needs no Example.

CASE II.

To meaſure any part of a Right Circle.

RULE.

1. Lay a Ruler from the Pole of the Right Circle, to the 2 Extremities of the Portion of the Right Circles, whoſe meaſure you'd know, and it will cut the primitive Circle in two Points.
2. Take the Diſtance of thoſe two Points in your Compaſſes, and apply it to your Line of Chords, and it will ſhew you its Meaſure.

EXAM-

Plate XI. *Fig.* II.

EXAMPLE.

I would know the Meaſure of the Portion of the Right Circle F E, a Ruler laid from its Pole D to E, cuts the primitive Circle in B; and laid from D to F, cuts the primitive Circle in *a*. Take *a* B in your Compaſſes, and apply it to your Line of Chords, and it ſhews you the number of Degrees, that E F repreſents, or its Meaſure.

The like for any other Portion great or ſmall, whether taken from the Center or not.

CASE III.

To Meaſure any Portion of an Oblique Circle.

RULE.

1. Find the Pole of the given Oblique Circle.
2. Lay a Ruler on the Pole to the two Extremities of the Portion, whoſe Meaſure is deſired.
3. The Ruler ſo laid, will cut the Primitive in two Points, whoſe Diſtance Meaſured on the Line of Chords, gives what is deſired.

I conceive it needleſs to give an Example, the Rule being ſo plain.

PROB. VIII.

To meaſure any Spheric Angle.

General Rule.

A Spheric Angle in the Projection, is meaſured by meaſuring that Arch of a great Circle, which is intercepted between the two Arches that contain the Angle at a Quadrant's Diſtance from the Angular Point; that is, the Angular Point being the Pole of the Arch, if ſuch an Arch be deſcribed in the Projection; If not meaſure the Diſtance between the two Poles of the

Circles containing the Angle; and that is the meaſure of the Angle required.

In this Problem are three Caſes.

1. When the Angular Point is in the Center of the primitive Circle.
2. When the Angular Point is in the Periphery of the primitive Circle.
2. When the Angular Point is within the Primitive Circle, but not in the Center.

CASE I.

To meaſure an Angle, when the Angular Point is in the Center.

RULE.

Such an Angle is meaſured by taking the Arch it ſubtends in your Compaſſes, and applying it to your Line of Chords.

CASE II.

To meaſure an Angle when the Angular Point is in the Periphery of the primitive Circle.

EXAMPLE.

Plate XIV.
Fig. III.

A B C D the Primitive Circle } given { E } their
BP D an Oblique Circle } { X } Centers.

'Tis required to Meaſure the Angle K B H.

Through X and E, draw the Diameter A C.

In this Diameter find x the Pole of the Oblique Circle B P D, the Diſtance E x or A P, meaſured by a Scale of half Tangents is the Meaſure of the Angle required.

After the ſame manner P E or C x, is the Meaſure of the Angle P B E.

CASE III.

To meaſure an Angle, when the Angular Point is within the primitive Circle, but not in the Center.

EXAMPLE.

ABCD the Primitive	Circle, and	E	their Centers given.
DPGB an Oblique		X	
LOGF an Oblique		Y	

'Tis required to meaſure the Angle F G B.

1. Through E and Y, and alſo through E and X, draw 2 Diameters, and in them find *y* and x, the Poles of the two Circles conſtituting the Angle, by *Prob.* I. *Caſe* III.
2. Lay a Ruler from G, the Angular Point to x and *y*, which ſhall cut the primitive Circle in *n* and *m*, the Diſtance *n m* taken in the Chords, is the meaſure of the Angle required.

The ſame Method is to be taken, if it were required to Meaſure the Angle B H K, conſtituted by a Right Circle and an Oblique, *viz.* A Ruler laid from H, to F and x, the length of the Arch in the Primitive Circle, intercepted between the Points cut by the Ruler, ſhall be the Meaſure of the Angle.

PROB. IX.

To draw a Parallel Circle.

DEFINITION.

Parallel Circles on the Globe, are ſuch which divide the Surface of the Globe into two unequal Parts, and are always Parallel to ſome great Circle. Parallels on the Projection are the Repreſentation of theſe.

In this Prob. are three Caſes; either,

1. Parallel to the Primitive Circle.
2. Parallel to a Right Circle.
3. Parallel to an Oblique Circle.

CASE I.

To draw a Circle parallel to the primitive Circle.

RULE.

This is done by taking the Tangent of half the Parallel's Diſtance from the Pole, and therewith (one Foot being placed in the Center) deſcribe a Circle and 'tis done.

EXAMPLE. *Plate* XII. *Fig.* IV.

A B C D, the primitive Circle, E its Center.

'Tis required to draw a Circle parallel to the primitive, and 30 *deg.* diſtance from it, that is, 60 *deg.* from the Pole.

Take E F equal to the Tangent of 30 *deg.* and with that diſtance, one foot being placed in E, deſcribe the Circle F G H M, and 'tis done.

CASE II.

To draw a Circle parallel to a Right Circle.

RULE.

1. From the Chords, lay off the Parallel's Diſtance from the Pole of the Right Circle given, both ways, marking the Points.
2. Take the Tangent of that Parallel's Diſtnace from the Pole, in your Compaſſes, and put one Foot in the two Marks before, Noted ſucceſſively,

ſtrike 2 Arches cutting one another; the Point of Interſection, ſhall be the Center of that Parallel.

Or thus, Croſs the Right Circle given with a Diameter at Right Angles, which let be produced; take the Secant of the Parallel's Diſtance from the Pole, and ſet it off in this Diameter, from the Center on that ſide the Circle ſhould be drawn, and it ſhall give the Center required.

Plate XII.

EXAMPLE.

Fig. IV.

A B C D the primitive Circle, E its Center,
B E D a Right Circle.

'Tis required to draw a Circle parallel to to the Right Circle B E D, at 40 *deg.* diſtance from it, or 50 *deg.* from its Pole C.

1. Lay off 50 *deg.* from C, to K and L.
2. Take the Tangent of 50 *deg.* and placing the foot of the Compaſſes, open'd to that diſtance, in L and K, and ſtrike 2 Arches cutting one another in Y.

Or the Secant of 50 *deg.* (the Parallels diſtance from the Pole) ſet off from E in the Diameter A C) will reach to Y, the Center requir'd.

CASE III.

To draw a Circle parallel to an oblique Circle.

RULE.

From the Scale of half Tangents, lay off the Parallel Diſtance from the Pole of the Oblique Circle given, both-ways, in that Diameter of the Primitive Circle, which paſſes through the Pole of the given Oblique Circle, then Note thoſe Marks; the Diſtance of theſe
Points

Points biſſected, gives the Center of the Parallel.

Plate XII. *Fig*. V.

EXAMPLE.

A B C D the Primitive } Circle, { E } their Centers.
D F B an Oblique } Circle, { X } their Centers.

'Tis required to draw a Parallel to the Oblique Circle, D F B, 40 *deg. diſtant from it, or* 50 *deg. diſtant from its Pole.*

1. Find G, the Pole of the given Oblique Circle D F B, by *Prob.* I. *Caſe* III.
2. Meaſure E G, by a Scale of half Tangents (or by a Scale of Tangents, doubling the Angle you find it) and ſuppoſe it to be 34 *deg.* then take the 34 *deg.*+50 *deg.*=84, and lay it down by a Scale of half Tangents, from E to *n*, and take 50 *deg.* —34 *deg.*=16; lay it off in the Line A C, from E to *m*, *m n* is the Diameter of this Parallel biſſect *m n* in *o*, *o* is the Center required. Or the Tangent of half 84, *viz.* 42 ſet off from E, will reach to *n*, and the Tangent of 8 *deg.* put off from E, will reach to *m*. Then with the Center *o*, and Radius *o m*, deſcribe the Parallel.

PROB. X.

To meaſure any Arch of a Parallel projected.

In this Problem there are three Caſes.

1. When 'tis Parallel to the Primitive Circle.
2. When 'tis Parallel to a Right Circle,
3. When 'tis Parallel to an Oblique Circle.

CASE I.

To meaſure any Arch of a Parallel to the primitive Circle.

RULE.

A Ruler laid from the Pole (or Center of the primitive Circle) to the equal Diviſions in the Periphery of the primitive Circle, ſhall cut the correſpondent Degrees in the Parallel.

CASE II.

To meaſure any Arch of a Parallel to a Right Circle.

Plate XII.
Fig. IV.

EXAMPLE.

Suppoſe I were to Meaſure or Divide the Parallel K I L, which is 50 *deg.* diſtant from C, the Pole of the Right *Circle* D E B.

RULE.

1. Deſcribe a *Circle* parallel to the *primitive Circle*, as far diſtant from the *Pole* in which the Eye is, as the parallel K I L, is from C, the Pole of the *Right Circle* D E B, 50 *deg.* and conſequently diſtant from the oppoſite Pole, in which the Eye is ſuppoſed 130 *deg.* Take the Semi-tangent of 130 *deg.* (or which is the ſame) the Tangent of 65 *deg.* and with your Compaſſes open'd to that Diſtance, and one Foot in the Center E, deſcribe a *Circle.*

2. A Ruler laid from C, to the equal Diviſions of that half of this Circle, which is parallel to the Semicircle B A D, ſhall cut the Parallel K I L, into its correſpondent Degrees.

Or a Ruler laid from A, the other Pole of the right Circle D E B, to the equal Degrees of that Semicircle which is parallel to B C D, ſhall likewiſe divide, or meaſure the parallel K I L.

CASE III.

To meaſure any Arch, or Parallel to an Oblique Circle.

EXAMPLE. *Plate* XII. *Fig.* V.

Suppoſe I were to meaſure, or divide the Parallel *m* R *n*, parallel to the Oblique Circle D F B, and diſtant from its Pole G, 50 *deg.* A Parallel to the Horizon muſt be deſcribed, as much diſtant from the Point or Pole in which the Eye is, *viz.* 50 *deg.* and conſequently diſtant from the oppoſite Pole, in which the Eye is 130 *deg.* take therefore the Tangent of 65 *deg.* or Semi-tangent of 130 *deg.* and deſcribe ſuch Circles as in the 2d Caſe. A Ruler laid to G, and the equal Diviſions of this Circle (parallel to the Primitive) ſo deſcribed, ſhall divide the Parallel *m* R *n*, into correſpondent Degrees or Parts.

Other Methods for doing the ſame thing, you'll find in what I have ſaid in the Theoretic part of *Stereographic Projection*, to which I refer you.

Note, I have omitted the drawing of theſe Parallels, mentioned in the 2d and 3d Caſe, ſince they would run beyond the Limits of the reſpective Figures, and becauſe it is ſo very eaſy to conceive them as drawn.

THE
Stereographic Projection
OF THE
Several Caſes in Right Angled SPHERIC TRIANGLES.

PROB. I.

The two Legs given, to find the two Oblique Angles and Hypothenuſe.

EXAMPLE.

GIven $$\left.\begin{array}{l}BA=30d.\\PA=25\end{array}\right\}\text{Required}\left\{\begin{array}{l}\text{Angles}\left\{\begin{array}{l}B\\P\end{array}\right.\\\text{Hypothenuse } BP.\end{array}\right.$$

CASE I.

Suppoſe the Leg A B be laid off in the Primitive Circle, then thro' A draw a great Circle at Right Angles to A B, which will be a Right Circle or Diameter; then with the Semi-tangent of 65 *deg.* the compliment of P A (or which is the ſame thing, with the Tangent of its half, *viz.* 32 *d.* 30′) one Foot of the Compaſſes being placed in the center, deſcribe a Circle parallel to the primitive, cutting the perpendicular A P in P. Laſtly thro' B and P draw a great Circle, and 'tis done.

Plate XIII.
Fig. I.
Caſe I.

To Meaſure the things required.

The $\left\{\begin{array}{l}\text{Hypothenuſe B P}\\ \text{Angles B, and P}\end{array}\right\}$ are meaſured by Prob. $\left\{\begin{array}{l}7\\ 8\end{array}\right.$

CASE II.

Suppoſe the Leg B A, *laid down in a Diameter, or Right Circle, the Point* B *being at the Center.*

The primitive Circle being drawn, and likewiſe the right Circle, in which you deſign to lay down B A; from the Center B, take the Tangent of 15 *deg.* or Semi-tangent of 30 *deg.* and lay it in the Diameter from B to A; then draw a great Circle, paſſing thro' the Poles of the Arc B A, and the point A, which will therefore be perpendicular to B A: In this Circle lay off A P=25 *deg.* by means of its Pole. Laſtly, draw a Diameter thro' P and B, and 'tis done.

Plate XIII.
Fig. I.
Caſe II.

To Meaſure the things required.

The $\left\{\begin{array}{l}\text{Hypothenuſe B P}\\ \text{Angles B, and P}\end{array}\right\}$ are meaſured by Prob. $\left\{\begin{array}{l}7\\ 8\end{array}\right.$

CASE III.

Suppoſe the Leg A B, *laid down in a Diameter, or Right Circle, the point* B *being not in the Center, but at any point aſſigned in that Diameter.*

1. By the help of a Scale of half Tangents, lay off B A, to repreſent 30 *deg.*

Plate XIII.
Fig. I.
Caſe III.

3. Then thro' the Poles of the Right Circle given, and the point A, deſcribe a great Circle, which is therefore perpendicular to B A, and therein meaſure A P=25 *deg.* by means of its *Pole.*

3. Laſtly, a great Circle ſtruck through the Points B and P, anſwers the *Problem.*

The Hypothenuſe and Angles P and B, are meaſured by the 7th and 8th *Prob.*

Note, That the Leg A B, may be laid down in any Oblique Circle, either from its Interſection with the primitive Circle, or from any point given therein, which I forbear to inſiſt upon, not doubting but the Reader, knowing what was done before, will eaſily apprehend.

PROB. II.

The Hypothenuſe, and one of the acute Angles given, to find the other Oblique Angle, and two Legs.

EXAMPLE.

$$\left.\begin{array}{l}\text{Given, P B} = 60d\\ \text{Angle, P} \;\;= 50\end{array}\right\}\text{to find}\left\{\begin{array}{l}\text{P A,}\\ \text{B A,}\\ \text{Angle B.}\end{array}\right.$$

There are ſeveral Varieties, I ſhall only Inſtance in Two.

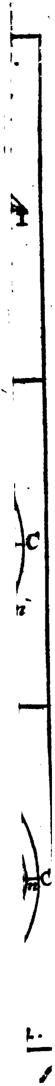
CASE I.

Where the Angle P *given, is placed in the Periphery of the primitive Circle.*

Plate XIII. 1. From the point P, deſcribe a Parallel to a Right Circle at 60 Degrees diſtance.
Fig. II.
Caſe I.

2. Draw a great Circle through P, making an Angle of 50 *deg.* with the primitive Circle, and cutting the parallel in B.

3. Through


A
C
n
m
C

3. Through the Center of the primitive and point B, draw a Diameter, cutting the primitive Circle in A, and 'tis done.

CASE II.

Where the Point P *is given at pleaſure.*

1. Through P, and the Center of the primitive Circle, draw a Diameter, or Right Circle. *Plate* XIII. *Fig.* II. *Caſe* II.
2. By *Prob.* 2d of *Stereog. Prob.* ſtrike a great Circle paſſing through the point P, and making an Angle with the Diameter drawn through P, of 50 *deg.*
3. By the help of the Pole of this Circle, meaſure PB=60.
4. Through the *Poles* of the Diameter paſſing thro' P and the Point B, ſtrike a great Circle cutting the Diameter in A, and 'tis done.

The parts required, may be meaſured by the help of the 7th and 8th *Prob.* of *Stereog. Prob.*

PROB. III.

A Leg with its adjacent Angle given to find the reſt.

EXAMPLE.

Given	P A . 50*d.*	Reqd.	P B,	*Plate* XIII.
	P . . 36		B A,	*Fig.* III.
			Angle B.	*Caſe* I.

I ſhall inſtance only in two Caſes.

1. Where the Leg P A, is projected in the primitive Circle do thus;

1. From P, lay off PA=50 *deg.* in the primitive Circle.
2. Through P, draw a great Circle, making an Angle of 36 *deg.*
3. Through A, draw a Diameter or Right Circle, and 'tis done.

2. Where the Leg A P is projected in the Diameter, the point P being in the Center.

Plate XIV.
Fig. I.

1. Draw a Diameter, and from the Center, lay off the Tangent of 25 *deg.* or Semitangent of 50, from the center P to A.
2. Draw another Diameter, making with the given Diameter, an Angle of 30 *deg.*
3. Thro' the *Poles* of the given Diameter, in which P A is projected, and point A, ſtrike a great Circle, cutting the 2d Diameter in B, then is P B A the Triangle required, whoſe parts may be meaſured by *Prob.* 7 and 8, of *Stereg. Prob.*

P R O B. IV.

The Hypothenuſe, and one Leg given, to find the reſt.

E X A M P L E.

Plate XIV.
Fig. II.
Caſe I.

Given B P. 46*d.* } Reqd. { B A,
P A 37 } { B and
{ P.

I ſhall take Notice only of two Caſes.

1. Where B is in the *Periphery* of the *primitive Circle*.
2. Where 'tis in the Center.

C A S E I.

1. Deſcribe a Parallel to a Right Circle, whoſe *Pole* B, is diſtant from B, 46 *deg.*
2. Deſcribe a Parallel to the primitive Circle, diſtant from the Center, by the Complement of P A, *viz.* 53 *deg.* which may cut the former Parallel in P.
3. Through the Center of the primitive Circle, and the point P, draw a Diameter cutting the primitive Circle in A.
4. Laſtly, Through B and P, deſcribe a great Circle, and 'tis done.

Plate 13.

Fig. 1.
Caſe 2d

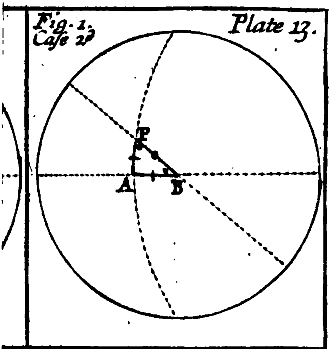


Fig 2d
Caſe 1.

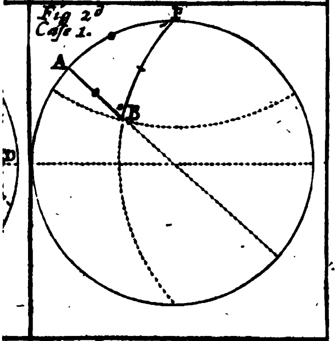
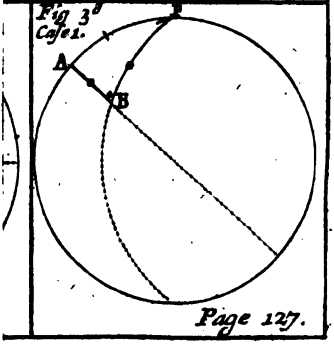


Fig 3d
Caſe 1.



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CASE II.

Plate XIV. *Fig.* II. *Caſe* II.

1. Draw a Diameter.
2. On the Center B, with the Semi-tangent of 46 *deg.* deſcribe a Circle parallel to the primitive Circle.
3. Deſcribe a *Parallel* to the firſt Diameter, diſtant from its *Pole* by the diſtance of the Complement of P A, which will interſect the *Parallel* to the primitive Circle P.
4. Through the Center of the primitive Circle, and point P, draw a Diameter.
5. Through the *Pole* of the firſt Diameter, and point P, draw a great Circle, and you ſhall have the Triangle B P A.

Whoſe unknown parts may be meaſured by *Prob.* 7, 8, of *Stereog. Prob.*

PROB. V.

A Leg with its oppoſite Angle given, to find the reſt.

EXAMPLE.

Plate XIV.	Given Angle B = 24*d.* 40′	Req.	A B,
Fig. III.	P A = 16. 17		B P,
			An. P.

1. Draw an Oblique Circle, making an Angle with the primitive, at the point B, of 24*d.* 40′, by *Prob.* 2. of *Stereog. Prob.*
2. With the Semi-tangent of 73*d.* 40′. the Complement of P A, one Foot being in the Center, draw a Circle parallel to the primitive, cutting the Oblique Circle in P.
3. Through the Center of the primitive Circle, and point P, draw a Diameter, or Right Circle, cutting the primitive Circle in A, and 'tis done.

The parts required, are meaſured by the 7th and 8th *Prob.* of *Stereog. Prob.*

PROB. VI.

Both Oblique Angles given, to find the reſt.

EXAMPLE.

Plate XIV.
Fig. IV.

$$\text{Given}\left.\begin{array}{l} B=46\,\mathit{deg.}\\ P=59\end{array}\right\}\text{Reqd.}\left\{\begin{array}{l}B\,P,\\ B\,A,\\ P\,A.\end{array}\right.$$

1. By *Prob.* 2. make the Angle B P A = 59 *deg.* by drawing the Oblique Circle P A.
2. Find the *Pole p*, of the Oblique Circle P A, by *Prob.* 1. *Caſe* 3. of *Stereog. Prob.*
3. Through *p*, the *Pole*, draw a great Circle, ſo that it may make with the primitive, an Angle of 46 *deg.* which will cut the Oblique Circle P A in A, and 'tis done.

The three things required, *viz.* A B, B P, P A, are meaſured by the 7th *Prob.* of *Stereog. Prob.*

THE PROJECTION OF *Oblique Spherical Triangles.*

PROB. VII.

Two Sides, and an Angle oppoſite to one of them given, to project the Triangle, and find the reſt.

EXAMPLE.

$$\text{Given}\ \left.\begin{array}{l} BP=34d.\ 50' \\ DP=25\ \ \ 30 \\ B=30\ \ \ 7 \end{array}\right\}\text{Reqd.}\left\{\begin{array}{l} P, \\ D, \\ BD. \end{array}\right.$$

Plate XIV. *Fig.* V.

Note, That this Caſe is Ambiguous, there being two Anſwers, but by the Projection, you may give both.

1. Make the Angle P B D=30*d.* 7′, by *Prob.* 2. *Caſe* 2. by drawing the Oblique Circle B D.
2. On the primitive Circle, lay off B P, 34 *deg.* 50.
3. Draw a parallel Circle diſtant from P, 25*d.* 30*m.* cutting the Oblique Circle in two points, D. D.
4. Through P, and each of the points D. D. draw great Circles, and you have two Triangles B P D, B P D, in one of which the Angle D is acute, in the other Obtuſe, &c.

The parts required, may be meaſured by *Prob.* 7 or 8, of *Stereog. Prob.*

P R O B. VIII.

Two Angles, and a Side oppoſite to one of them given, to project the Triangle, and find the reſt.

E X A M P L E.

Plate XV. Fig. I.

Given B P = 60*d.* 10′ Reqd. B D,
B = 40 00 P D,
D = 50 00 P.

1. By *Prob.* 2. of *Stereog. Prob.* make the Angle P B D = to 40 *deg.* by drawing the Oblique Circle B P.
2. Draw a parallel to a Right Circle, diſtant from B, 60 *deg.* cutting the Oblique Circle in P, by *Prob.* 9. of *Stereog. Prob.*
3. Through P, draw a great Circle, ſo as to make an Angle of 50 *deg.* as D P; and 'tis done.

The things required, are meaſured by *Prob.* 7 and 8, of *Stereog. Prob.*

P R O B. IX.

Two Legs with an Angle included, to project the Triangle, and find the reſt.

E X A M P L E.

Plate XV. Fig. II.

Given B P = 60*d* Reqd. P D,
B = 40 P,
P D = 87 30′ D.

1. Draw an Oblique Circle B D, making an Angle with the primitive Circle = 40 *deg.* by *Prob.* 2. of *Stereog. Prob.*

2. Make BP = 60 *deg.* and by help of the *Pole* of the Oblique Circle BD, make BD = 87 *deg.*
3. Through P and D, draw a great Circle, and 'tis done.
4. The parts required, are measured by *Prob.* 7 and 8, of *Stereog. Prob.*

PROB. X.

Two Angles and a Side included given, to project the Triangle and find the rest.

EXAMPLE.

Given B 40*d*		PD,		
P 118 06'	Reqd.	BD,	*Plate* XV.	
BP 60 10		D.	*Fig.* III.	

1. Make BP = 60 *deg.* 10', by *Prob.* 6, *Case* 1. of *Stereog. Prob.*
2. Make the Angle BPD = 118 *deg.* 06', by drawing the Oblique Circle PD, and at the point B, make the Angle PBD = 40 *deg.* by *Prob.* 2. which two Oblique Circles will intersect in D, and 'tis done.

The parts required are measured by *Prob.* 7 and 8. of *Stereog. Prob.*

PROB. XI.

Three Sides being given, to project the Triangle, and find the Angles.

EXAMPLE.

Given BP = 60*d* 10'	Reqd.	B,	
DP = 46 12	the	P,	*Plate* XV.
BD = 87 30	Angles	D.	*Fig.* IV.

On

1. On the primitive Circle, make B D=87*d.* 20′ the greater given Side, by help of a Line of *Chords.*
2. Draw a parallel Circle at 60*d.* 10′ diſtance from B, and likewiſe in another at 46 *d.* 12′ diſtance from D, cutting each other in the point P.
3. Through B and P, and likewiſe through D and P, draw a great Circle and 'tis done.

Note, That in the Triangle H I K, made by the Circle's paſſing through the Poles of the ſides B D, B P, and D P; H I is the meaſure of the Angle B, H K the meaſure of the Angle D, K I the Supplement of the meaſure of the Angle P, which may be meaſured by *Prob.* 7. of *Stereog. Prob.*

PROB. XII.

Three Angles being given, to make a Triangle, and conſequently to meaſure the Sides.

RULE.

Plate XV.
Fig. IV.

With the three Angles given, as if they were ſides, project the Triangle B D P.

Then meaſure the ſides of the Triangle H I K, made by three Arcs paſſing through the Poles of the ſides D B. B P and D P, and you have the Angles of the Triangle required; ſave that the greateſt ſide is the Supplement of the greateſt Angle in the Triangle B D P, and conſequently of the greateſt ſide in the Triangle required.

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